CONTROL OF INFINITE DIMENSIONAL SYSTEMS USING FINITE DIMENSIONAL TECHNIQUES: A SYSTEMATIC APPROACH

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Submitted to the Department of

ELECTRICAL ENGINEERING AND COMPUTER SCIENCE

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

August 1990

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Report Documentation Page

Form Approved
OMB No. 0704-0188

1. REPORT DATE **AUG 1990**	2. REPORT TYPE	3. DATES COVERED **00-00-1990 to 00-00-1990**

4. TITLE AND SUBTITLE **Control of Infinite Dimensional Systems Using Finite Dimensional Techniques: A Systematic Approach**	5a. CONTRACT NUMBER 5b. GRANT NUMBER 5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER 5e. TASK NUMBER 5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **Massachusetts Institute of Technology,77 Massachusetts Avenue,Cambridge,MA,02139**	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S) 11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES

14. ABSTRACT

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**Same as Report (SAR)**	**135**	

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

Control of Infinite Dimensional Systems using Finite Dimensional Techniques: A Systematic Approach

by: Armando Antonio Rodriguez

Submitted to the Department of Electrical Engineering and Computer Science on August 15, 1990 in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

In this thesis, the problem of designing finite dimensional controllers for infinite dimensional single-input single-output systems is addressed. More specifically, it is shown how to systematically obtain near-optimal finite dimensional compensators for a large class of scalar infinite dimensional plants. The criteria used to determine optimality are standard $\mathcal{H}^\infty$ and $\mathcal{H}^2$ weighted sensitivity and mixed-sensitivity measures.

Unlike other approaches which appear in the literature, the approach taken here avoids solving an infinite dimensional optimization problem to get an infinite dimensional compensator and then approximating to get an appropriate finite dimensional compensator. Rather than this *Design/Approximate* approach, we take an *Approximate/Design* approach. In this approach one starts with a "good" finite dimensional approximant for the infinite dimensional plant and then solves a finite dimensional optimization problem to get a suitable finite dimensional compensator. Traditionally, however, this approach has not come with any guarantees.

The key difficulties which have arisen can be attributed to the fact that these measures are sometimes not continuous with respect to plant perturbations, even when the uniform topology is imposed. Moreover, even if they were, it is a known fact that many interesting infinite dimensional plants can not be approximated in the uniform topology on $\mathcal{H}^\infty$ (e.g. a delay). Also, it must be noted that the concept of a "good" approximant, in the context of feedback design, has never been rigorously formulated.

The goal and main contribution of this research endeavour has been to resolve these difficulties. It is shown that given a "suitable" finite dimensional approximant for an infinite dimensional plant, one can solve a "natural" finite dimensional problem in order to obtain a near-optimal finite dimensional compensator. Moreover, very weak conditions are presented to indicate what a "good" approximant is.

In addition, we show that the optimal performance for a large class of $\mathcal{H}^\infty$ design paradigms can be computed by solving a sequence of finite dimensional eigenvalue/eigenvector problems rather than the typical infinite dimensional eigenvalue/eigenfunction problems which appear in the literature. Analogous results are presented for a large class of $\mathcal{H}^2$ paradigms.

In summary, the approach taken here allows one to forgo solving "complex" infinite dimensional problems and provides rigorous justification for some of the approximations that control engineers typically make in practice.

Thesis Supervisor: Dr. Munther A. Dahleh
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Acknowlegdements

First and foremost, I would like to thank my thesis supervisor, Professor Munther Dahleh. He has always offered infinite support, both technical and otherwise. His guidance, patience, and deep concern have been very much appreciated. I have been extremely fortunate to have had him as my thesis supervisor. I cannot thank him enough. I feel lucky not only to have had the opportunity to learn from him, as a student, but also to have associated with him as a person. Although I will see him at conferences in the future, I will still miss him a great deal.

I would also like to thank my thesis committee members Dr. Gunter Stein and Professor Sanjoy Mitter for their helpful consultation throughout the course of this research. Every session with them provided perspective and great insight. It was a pleasure and an honor to work with them.

In addittion, I must thank Professor Michael Athans, Ignacio Diaz-Bobillo, Joel Douglas, Professor David Flamm, Dr. Petros Kapasouris, Sanjeev Kulkarni, Joseph Lutsky, Dr. Dragan Obradović, Professor Jason Papastavrou, Dr. Tom Richardson, Professor Brett Ridgely, Professor Jeff Shamma, Petros Voulgaris, Jim Walton. All have contributed in some way or form.

The staff at LIDS was always helpful and supportive. I would particularly like to thank Fifa Monserrate. She was always kind, warm, and cheerful.

I must also give great thanks to AT&T Bell Laboratories for their most generous fellowship. I would like to specifically thank my mentor at the labs, Dr. Sid Ahuja. He has always been supportive.

Finally, I must give special thanks to my father and my brothers David and Raul for the love, encouragement, and support that they have given me over the years. I would also like to thank my dear friend Ted Hernandez who unfortunately passed away during this last year. I owe him so much.

This research was conducted at the M.I.T Laboratory for Information and Decision Systems. It has been supported by AT&T Bell Laboratories, by the Center for Intelligent Control Systems under the Army Research Office grant DAAL03-86-K-0171, and by NSF grant 8810178-ECS.

To my father.

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Chapter 1

Introduction & Overview

1.1 Motivation

During the 1980's, the need to address the control of large scale flexible structures has increased immensely. Although this need has arisen primarily from SDI and aerospace applications, it has also been fueled by complexity issues in power distribution and other areas. Because of these driving forces, the problem of designing feedback control systems for infinite dimensional systems has received considerable attention during the past decade. Researchers have endeavoured to develop systematic design procedures. In this thesis such systematic design procedures are presented for a large class of infinite dimensional systems. More specifically, it is shown how to obtain near-optimal finite dimensional compensators for $\mathcal{H}^\infty$ and $\mathcal{H}^2$ weighted sensitivity and mixed-sensitivity performance criteria. The approach taken in this thesis is now motivated.

Throughout the thesis, it shall be assumed that the designer has been given a single-input single-output infinite dimensional plant[1] and a performance measure. In the spirit of the seminal work of Zames [59], it will also be assumed that the performance measure has been posed as an infinite dimensional optimization problem. The goal then is to design a near-optimal finite dimensional compensator. The finite dimensionality, of course, is a typical "real-world" implementation constraint. This thesis addresses the problem of designing near-optimal finite dimensional compensators. Two approaches to this problem have appeared in the literature.

The first approach we call the *Design/Approximate* approach. In this approach an optimal infinite dimensional compensator is designed by solving, if possible, the infinite dimensional optimization problem. The optimal compensator is then approximated by a finite dimensional compensator. This approach will not be considered in the sequel.

The second approach we call the *Approximate/Design* approach. In this approach the infinite dimensional plant is approximated by a sequence of finite dimensional plants. We then solve a sequence of "natural" finite dimensional problems in which we simply substitute the finite dimensional approximants for the infinite dimensional plant in the original optimization problem. This sequence of finite dimensional problems generate a sequence of finite dimensional compensators which, ideally, will be near-optimal as the plant approximants get "better". Traditionally, however, no such guarantees have been shown.

The key difficulties which have arisen can be attributed to the fact that these performance measures are often not continuous with respect to plant perturbations, even when the uniform topology is imposed. In this thesis, these difficulties are resolved; guarantees are provided.

[1] The system to be controlled is called the *plant*.

The primary motive behind the *Approximate/Design* approach taken in this work has been that of finding near-optimal finite dimensional compensators for scalar infinite dimensional systems. Other motives can be listed as follows.

(1) Some infinite dimensional models are too complex. It is often very difficult to gain intuition from them. Designing controllers based on such models often requires advanced mathematical machinery and new software. It follows naturally to ask: What would be a "good" finite dimensional approximant? Such an approximant should give immediate insight. To design controllers which are based on such an approximant usually requires little mathematical sophistication. Moreover, much software exists for such a finite dimensional approach. The above question raises the following question: What information about the infinite dimensional plant do we really need in order to achieve the control objective? The approach taken in this thesis attempts to shed light on the above questions.

(2) Some design procedures result in compensators which are infinite dimensional. Such compensators may be difficult, if not impossible to implement. The following natural question thus arises: How can we obtain a finite dimensional compensator which is suitable? The *Approximate/Design* approach taken in this thesis addresses this question directly.

(3) Often, in the early stages of system planning and design, it is necessary to estimate achievable system performance. Such an estimate could be used for system reconfiguration and enhancement. It thus follows that efficient computational tools to obtain such performance information would be extremely valuable to system designers. By taking an *Approximate/Design* approach, one addresses such computational issues indirectly.

1.2 Related Work & Previous Literature

The problem of designing compensators for infinite dimensional plants has recieved considerable attention during the past decade. Some relevant works are [1], [6]-[10], [13]-[20], [25]-[26], [31]-[38], [40]-[41], [45], [47], [51]-[53], [57]-[63]. We now give a chronological summary of some of this work.

1950's

The works of [9], [10], and [32] address approximation issues. Real-rational approximation methods are presented. The methods are based on "open loop" ideas and not on closed loop performance criteria.

1960's

Much of the technical issues which arise in todays $\mathcal{H}^\infty$ model matching approach to control synthesis, were addressed in the famous paper of [51]. In this paper the author solves various interpolation problems in $\mathcal{H}^\infty$. The paper contains the *commutant lifting theorem* and shows how one can construct norm preserving $\mathcal{H}^\infty$ dilations for various operators. This work has been the cornerstone of many approaches/solutions which have appeared for the $\mathcal{H}^\infty$ sensitivity and mixed-sensitivity problems.

1970's

The Hankel matrix approximation problem is solved in [1]. This paper has also tremendously influenced the $\mathcal{H}^\infty$ model matching approach which is present everywhere in the control literature.

At the heart of todays model matching approach to control is the ability to parameterize all internally stabilizing compensators for a given plant. Such a parameterization was done in [58] for finite dimensional multivariable linear time invariant plants.

An algebra of transfer functions for distributed linear time invariant systems is presented in [4]. Elements in the fraction field of this algebra possess coprime factorizations over the algebra. The work of [58] can thus be used to parameterize the set of all internally stabilizing controllers for plants which lie within the fraction field of the algebra.

1981

In [59], the author posed what is now referred to as the weighted $\mathcal{H}^\infty$ sensitivity control problem. One objective of this seminal paper was to formulate control problems as optimization problems in an attempt to systematize control system design. It was argued and shown that this frequency domain approach is natural to handle unstructured uncertainty.

1982

The work of [21] gives a very nice solution for scalar weighted $\mathcal{H}^2$ sensitivity and mixed-sensitivity problems.

1983

In [60], the authors use duality and interpolation theory to solve the weighted $\mathcal{H}^\infty$ sensitivity problem for real-rational scalar plants. A fundamental motive for this work was to replace the heuristic aspects of classical design by an explicit mathematical theory.

1984

The commutant lifting theory of [51] is used in [22] to obtain an upperbound for the optimal weighted $\mathcal{H}^\infty$ sensitivity associated with scalar finite dimensional systems. The problem of achieving a small sensitivity over a specified frequency band is also addressed. The effects of non-minimum phase zeros is discussed.

A solution to the scalar weighted $\mathcal{H}^\infty$ mixed-sensitivity problem is presented in [55]. Here the mixed-sensitivity criterion used is that which penalizes the sensitivity and the complementary sensitivity transfer functions.

1985

Necessary and sufficient conditions for the existence of finite dimensional compensators for delay systems are presented in [31]. Moreover, it is shown that a stabilizable delay system can always be stabilized using a finite-dimensional compensator.

1986

A method for constructing finite dimensional compensators which stabilize infinite dimensional systems with unbounded input operators is presented in [6]. Applications to retarded and partial differential equations are considered.

In [7], the authors present a method for contructing robustly stabilizing finite dimensional compensators for a class of infinite dimensional plants.

The commutant lifting ideas in [51] are used by [14] to solve the weighted $\mathcal{H}^\infty$ sensitivity problem for the case where the plant is a product of a delay and a real-rational scalar function. Fairly general real-rational weighting functions are considered. The optimal sensitivity and compensator are computed in various situations. The implementation of the optimal infinite dimensional compensator is also discussed.

In [18], the authors also solve a weighted $\mathcal{H}^\infty$ sensitivity problem using the work of [51]. Here the plant is a delay and the weighting function is first order and strictly proper.

1987

The ideas presented in [14] are expanded upon in [15].

The results of [18] are extended in [19] to more general delay systems. The plant is assumed to be the product of a delay and a scalar real-rational transfer function with no poles or zeros on the imaginary axis. The weight is assumed to be an $R\mathcal{H}^\infty$ function which is invertible in $\mathcal{H}^\infty$. The interaction between delays and non-minimum phase zeros is also discussed.

A solution to the weighted $\mathcal{H}^\infty$ sensitivity problem is presented in [20] for arbitrary scalar distributed plants. An explicit formula is given for the optimal sensitivity. The existence and uniqueness of the optimal compensator is discussed.

In [63], the authors show how the optimal weighted $\mathcal{H}^\infty$ sensitivity for a delay can be computed by solving a two point boundary value problem. Here the weight can be any $R\mathcal{H}^\infty$ function.

The uniform approximation of a class of delay systems by means of partial fraction expansions is investigated in [62]. Nuclear systems are discussed.

1988

Krein space theory is used to tackle the $\mathcal{H}^\infty$ mixed-sensitivity problem in [13]. Here the mixed-sensitivity criterion considered penalizes the sensitivity and complementary sensitivity functions.

In [16], the authors expand upon the implementation issues presented in [14].

The problem of uniformly approximating delay systems is considered in [45]. Condition for nuclearity are given.

In [25], the authors show how to construct real-rational approximants for nuclear systems. More specifically, it is shown that for this class of systems, balanced or output normal realizations always exist and their trucncations converge to the original system in various topologies. Various error bounds are given.

An iterative procedure for constructing near-optimal infinite dimensional compensators for a class of infinite dimensional plants is presented in [57]. The optimality criterion is an $\mathcal{H}^\infty$ sensitivity criterion. The method presented assumes that the weighting function is strictly proper. In such a case the corresponding Hankel operator is compact.

The computation of the essential spectra of certain Hankel-Toeplitz operator pairs is crucial in the solution of $\mathcal{H}^\infty$ sensitivity and mixed-sensitivity problems. This is particularly important when infinite dimensional systems are involved. Such a computation is given in [61]. Calkin algebra techniques are used to obtain the results.

1989

In [8], the author addresses the control of infinite dimensional systems which belong to the algebra presented in [4]. Systems which are of the Pritchard-Salamon class are also addressed.

In [12], the authors present state space formulae to solve a myriad of finite dimensional $\mathcal{H}^2$ and $\mathcal{H}^\infty$ control problems.

An FFT-based algorithm for approximating infinite dimensional systems is presented in [28].

When approximating infinite dimensional systems by finite dimensional approximants, the rate at which the approximants converge is very important. Such convergence rate results are given in [26] for certain approximants and infinite dimensional systems. Pade approximations of delays, for example, are discussed.

An $\mathcal{H}^\infty$ mixed-sensitivity problem for a flexible Euler-Bernoulli beam is solved in [34]. The solution relies on the techniques used in [41].

In [36], the author uses Laguerre series to approximate certain infinite dimensional systems. Various error bounds are given.

Skew Toeplitz techniques are used in [40], to solve various control problems for infinite dimensional systems. The $\mathcal{H}^\infty$ mixed-sensitivity problem is addressed.

In [41], the authors reveal the structure of suboptimal $\mathcal{H}^\infty$ controllers for distributed plants.

In [52], the author shows that fraction field of $\mathcal{H}^\infty$ is a Bezout domain. That implies that plants in the fraction field of $\mathcal{H}^\infty$ possess coprime factorizations over $\mathcal{H}^\infty$. This, then allows us to paramerize all stabilizing compensators for such plants using the ideas of [58].

1990

A solution to the $\mathcal{H}^\infty$ mixed-sensitivity problem is presented in [17]. Here, very general distributed plants and irrational weighting functions are treated.

$\mathcal{H}^\infty$ mixed-sensitivity techniques are used in [42] to control unstable infinite dimensional plants. Skew Toeplitz methods are used to derive the controllers.

In [53], the author studies the continuity properties of various $\mathcal{H}^p$ problems. In particular, it was shown that $\mathcal{H}^\infty$ and $\mathcal{H}^2$ problems, in general, are discontinuous functions of the plant, even when the $\mathcal{H}^\infty$ topology is used. This paper contains many of the ideas presented in this thesis. It does not, however, address control design. It only addresses certain well-possedness issues which arise in certain optimization problems. We also would like to point out that although elements of this work was initially published in 1987, in a conference proceedings, it did not come to our attention until March 1990.

1.3 Contributions of Thesis

In this thesis, the *Approximate/Design Problem* is rigorously formulated. A solution is provided for $\mathcal{H}^\infty$ and $\mathcal{H}^2$ weighted sensitivity and mixed-sensitivity performance criteria. These are the main contributions of the thesis.

More specifically, it is shown that given a "good" finite dimensional approximant for an infinite dimensional plant, one can solve a "natural" finite dimensional problem in order to obtain a near-optimal finite dimensional compensator. Conditions are given which precisely quantify the notion of a "good" finite dimensional approximant.

Given this, the contributions of the thesis can be concretely stated as follows.

(1) A method for constructing near-optimal finite dimensional controllers for a large class of infinite dimensional scalar plants is presented. Stable and unstable plants can be handled. Much software exists to support the necessary computations. The method is thus immediately implementable by practicing engineers.

(2) The same method can be used to construct near-optimal infinite dimensional controllers. One can then "directly" obtain near-optimal finite dimensional compensators by approximating the infinite dimensional controllers. This approach, however, goes against the spirit of the thesis since it requires that we solve an infinite dimensional optimization problem.

(3) The most commonly used design criteria have been addressed; i.e. $\mathcal{H}^\infty$ and $\mathcal{H}^2$ weighted sensitivity and mixed-sensitivity problems. Again, much software exists to support our finite dimensional approach to these paradigms.

(4) For a large class of $\mathcal{H}^\infty$ weighted sensitivity/mixed-sensitivity problems, the *optimal performance* can be easily computed by solving a sequence of finite dimensional eigenvalue/eigenvector

problems rather than the typical infinite dimensional eigenvalue/eigenfunction problems which appear in the literature. The methods presented apply even in situations where the associated Hankel/Toeplitz operators are non-compact.

Such information can be used to determine fundamental performance limitations for a given system; e.g. best possible $\mathcal{L}^2$ disturbance rejection, robustness, etc. It can thus be used to guide designers (engineers, pilots, etc.) during the initial stages of system development, design, and configuration.

Analogous results are presented for the $\mathcal{H}^2$ design criteria considered.

(5) The thesis sheds light on such issues as what a "good" finite dimensional approximant is and hence on what information is needed in order to achieve a particular control objective. It is shown that appproximations should be based on the control objective; not on open loop intuition. Moreover, it is shown that open loop intuition can often be quite misleading. These ideas have potential implications in such fields as system identification and decentralized control.

1.4 Organization of Thesis

The remainder of this thesis is organized as follows.

In Chapter 2, notation and results from complex variable and approximation theory are presented. The function spaces $\mathcal{H}^\infty$ and $\mathcal{H}^2$ are defined and discussed. Various notions of convergence are presented.

In Chapter 3, results from algebraic system theory are presented; e.g. the Youla parameterization and the Corona theorem.

In Chapter 4 three problems are formulated. They are the *$\mathcal{N}$-Norm Approximate/Design J-Problem*, the *$\mathcal{N}$-Norm Purely Finite Dimensional J-Problem*, and the *$\mathcal{N}$-Norm Loop Convergence J-Problem.* In the sequel, the $\mathcal{N}$ will represent $\mathcal{H}^\infty$ and $\mathcal{H}^2$ norms. The J will represent sensitivity and mixed-sensitivity performance criteria.

The *$\mathcal{N}$-Norm Approximate/Design J-Problem* addresses the problem of finding near-optimal finite dimensional compensators. The *$\mathcal{N}$-Norm Purely Finite dimensional J-Problem* addresses the issue of computing the *optimal performance* using finite dimensional techniques. The *$\mathcal{N}$-Norm Loop Convergence J-Problem* addresses the question of what additional properties are exhibited by designs based on the *Approximate/Design* approach advocated in the thesis. The above three problems are considered in the sequel for $\mathcal{H}^\infty$ and $\mathcal{H}^2$ weighted sensitivity and mixed-sensitivity performance criteria.

In Chapter 5, the $\mathcal{H}^\infty$ *Model Matching Problem* is defined and discussed. It is shown how near-optimal solutions can be constructed. The results presented here are exploited heavily in subsequent chapters on $\mathcal{H}^\infty$ design.

In Chapter 6, the focus is on designing near-optimal finite dimensional compensators based on $\mathcal{H}^\infty$ sensitivity design criteria. More specifically, in this chapter a solution is presented to the $\mathcal{H}^\infty$ *Approximate/Design Sensitivity Problem*, the $\mathcal{H}^\infty$ *Purely Finite Dimensional Sensitivity Problem*, and the $\mathcal{H}^\infty$ *Loop Convergence Sensitivity Problem.* For simplicity, stable and unstable plants are treated separately.

In Chapter 7, the focus is on designing near-optimal finite dimensional compensators based on $\mathcal{H}^\infty$ mixed-sensitivity design criteria. More specifically, in this chapter a solution is presented to the $\mathcal{H}^\infty$ *Approximate/Design Mixed-Sensitivity Problem*, the $\mathcal{H}^\infty$ *Purely Finite Dimensional Mixed-Sensitivity Problem*, and the $\mathcal{H}^\infty$ *Loop Convergence Mixed-Sensitivity Problem.*

In Chapter 8, the focus is on designing near-optimal finite dimensional compensators based

on $\mathcal{H}^2$ sensitivity/mixed-sensitivity design criteria. More specifically, in this chapter a solution is presented to the $\mathcal{H}^2$ *Approximate/Design Sensitivity Problem*, the $\mathcal{H}^2$ *Purely Finite Dimensional Sensitivity Problem*, and the $\mathcal{H}^2$ *Loop Convergence Sensitivity Problem.* The analogous mixed-sensitivity problems are also addressed. The features which distinguish the $\mathcal{H}^2$ case from the $\mathcal{H}^\infty$ case are highlighted.

Finally, Chapter 9 summarizes the results of the thesis and suggests possible directions for future research.

Chapter 2

Mathematical Preliminaries: Notation & Function Theory

2.1 Introduction

In this chapter we establish notation to be used throughout the thesis. Some essential mathematical results are also presented. More specifically, the normed linear spaces $\mathcal{H}^2$ and $\mathcal{H}^\infty$ are defined and discussed. Various notions of convergence in $\mathcal{H}^\infty$ are presented; e.g. uniform and compact convergence. Results from $\mathcal{H}^\infty$ approximation theory are also presented. Examples are given to illustrate some of the ideas. Most of the material presented in this chapter can be found in [2], [5], [30], [33], [35], [44], [49], [56] [1].

2.2 Some Notation & Definitions

Throughout the thesis, we will use the symbols C, R, and Z to denote the complex, real, and integer numbers, respectively. C_e and R_e will be used to denote the extended complex and real numbers. The open right and left half complex planes will be denoted C_+ and C_-. R_+ and Z_+ will be used to denote the non-negative real numbers and positive integers. $|(\cdot)|$ and $\overline{(\cdot)}$ will denote the magnitude (modulus) and complex conjugate of the complex quantity $(\cdot)$. $\theta_{(\cdot)}$ and $\angle(\cdot)$ will be used to denote the phase angle of the complex quantity $(\cdot)$. The symbol j will be used to denote the purely imaginary number $\sqrt{-1}$.

The greek letter ϵ shall always be used in proofs to denote a given strictly positive, but arbitrarily small, quantity. With this convention we can avoid the excess verbage "given $\epsilon > 0$, however small,...".

Given sets A and B, $A \subset B$ and $A \subseteq B$ will be used to denote strict and non-strict containment of A within B. The symbol A/B will denote the set of points in A which are not in B. $A \cup B$ and $A \cap B$ will denote the *union* and *intersection* of the sets, respectively.

In the mathematics literature the *characteristic function of a set S* is defined as that function which is unity on S and zero elsewhere. It is typically denoted X_S. Throughout the thesis we shall predominantly work on the imaginary axis. This motivates the following definition which we introduce strictly for notational economy.

[1]Extra material has been included in this chapter in order to make the thesis self-contained and to facilitate future addendums.

Definition 2.2.1 (Characteristic Function)
Let S denote a subset of the extended real numbers. In what follows, the map $X_S : jS \to \{0,1\}$ will denote the *characteristic function* of the set jS; i.e.

$$X_S(j\omega) \stackrel{\text{def}}{=} \begin{cases} 1 & \omega \in S; \\ 0 & \text{elsewhere.} \end{cases}$$

■

Convention 2.2.1 (Transform Pairs)
We shall often use $f(t)$ or f and $F(s)$ or F, to denote *Laplace, Fourier, Plancherel transform* pairs. Here f (lowercase) will denote a time function and F (uppercase) will denote its transform. The interpretation should be clear from the context.

■

Convention 2.2.2 (Lebesgue Integral)
All integrals in this thesis shall be assumed to be *Lebesgue integrals* [49], unless otherwise stated. Any measure-theoretic statements which appear are made with respect to Lebesgue's measure unless otherwise stated [2].

■

Let $f(t)$ denote a real-valued Lebesgue measurable function defined on a set $S \subset R$.

Definition 2.2.2 (Support of a Function)
The *support* of f, denoted *suppf*, is defined to be the closure of the set where f takes on non-zero values; i.e.

$$suppf \stackrel{\text{def}}{=} closure\{t \in S \mid f(t) \neq 0\}.$$

■

Definition 2.2.3 (Essential Supremum of a Function)
The *essential supremum* of f on S is defined as follows

$$ess\sup f \stackrel{\text{def}}{=} \inf_M \{\ M \in R_e \mid measure(\ \{t \in S \mid f(t) \geq M\}\ ) = 0\ \}.$$

Here *measure*$(\cdot)$ denotes the Lebesgue measure of the set $(\cdot)$.

■

Definition 2.2.4 (Convolution of Functions)
Given any two time functions f and g with support on R, their *convolution* will be denoted $f * g$ and defined as follows

$$(f * g)(t) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau.$$

[2]Measure theoretic arguments shall be kept to a minimum throughout the thesis for added simplicity and brevity.

∎

Throughout the thesis we will deal with *normed linear spaces.* $||f||_{(\cdot)}$ will be used to denote the norm of a function f belonging to the normed linear space $(\cdot)$.

Definition 2.2.5 (Isometry, Isomorphism)
Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be normed linear spaces. A norm preserving linear operator from $\mathcal{N}_1$ to $\mathcal{N}_2$ is called an *isometry.* Such operators are necessarily injective (one-to-one). They need not be surjective (onto). $\mathcal{N}_1$ and $\mathcal{N}_2$ are said to be *isomorphic* if there exists a bijective bounded linear operator from $\mathcal{N}_1$ to $\mathcal{N}_2$ whose inverse is also bounded (cf. definition 2.8.1). The operator is called an *isomorphism.* A surjective isometry is an isomorphism. We call such an operator an *isometric isomorphism.*

∎

In what follows we shall deal with the normed linear spaces $\mathcal{H}^2$ and $\mathcal{H}^\infty$. They shall be defined shortly.

2.3 Single-Valued Complex Functions

In this section we present standard definitions and results from the theory of complex functions of a single complex variable. Let $F(s)$ or F, denote a complex-valued function of a single complex variable s. We assume all such functions to be single-valued unless otherwise stated. Let s_0 denote any point in the finite complex plane.

Definition 2.3.1 (Domain and Boundary)
A *domain* is a non-empty open connected subset of the extended complex plane. The *boundary* of a domain $\mathcal{D}$ shall be denoted $\partial\mathcal{D}$.

∎

The open right half plane is a domain.

Definition 2.3.2 (Domain of Definition)
The *domain of definition* of a single-valued complex function is a domain in the complex plane over which the function is defined.

∎

In the sequel, the domain of definition of a function will be ascertainable from the context. It will usually be the region of convergence of the function when viewed as the Laplace Transform of a time function.

Definition 2.3.3 (Analytic and Entire Functions)
We say that F is *analytic* at the point s_0, if F is differentiable at all points in some open neighborhood of s_0. We say that F is analytic within a domain $\mathcal{D}$, if it is analytic at each point within $\mathcal{D}$. If F is analytic within C, then we say that F is an *entire* function.

∎

Some authors use the term *holomorphic* rather than *analytic.*

Proposition 2.3.1 (Taylor Series)
Let F be analytic at the point $s = s_0$. Given this, there exists a neighborhood

$$N(s_0, R) \stackrel{\text{def}}{=} \{s \in C \mid |s - s_0| < R\}$$

about s_0, and a sequence $\{a_m\}_{m=0}^{\infty}$, such that

$$F(s) = \sum_{m=0}^{\infty} a_m (s - s_0)^m$$

for all $s \in N(s_0, R)$. Moreover,

$$a_m = \frac{F^{(m)}(s_0)}{m!}$$

and the series converges uniformly on all compact subsets within the neighborhood. We shall refer to this representation for F as the *Taylor series expansion* for F about s_0.

∎

The *radius of convergence*, of the Taylor series for F about s_0 is equal to the distance from s_0 to the nearest singularity of F (cf. definition 2.3.7). It is given by *Hadamard's formula*

$$R_0 = \frac{1}{\lim_{n\to\infty} \sup \sqrt[n]{|a_m|}}.$$

Definition 2.3.4 (Zero of a Function)
We say that s_0 is a *zero* of F, if

$$\lim_{s \to s_0} F(s) = 0$$

along any path in the *domain of definition* of the function F.

∎

Definition 2.3.5 (Multiplicity of a Zero for Analytic Functions)
Let F be analytic at $s = s_0$. Also, let s_0 be a zero of F. We say that s_0 has *multiplicity* $m \in Z_+$, if there exists a function G such that for all s in a neighborhood of s_0,

$$F(s) = (s - s_0)^m G(s)$$

where $G(s_0) \neq 0$ and G is analytic within the neighborhood.

∎

An integer m and a function G can always be found for any single-valued function F. In this sense, all single-valued functions exhibit polynomial behavior near a zero. It must be noted that for single-valued functions obtained from multi-valued functions, the situation is different. Such functions shall be discussed in the next section.

Definition 2.3.6 (Roll-off)
We shall say that F *rolls-off* if ∞ is a zero of F.

∎

Definition 2.3.7 (Singularity, Isolated Singularity)
If F is not analytic at s_0, then we say that $s = s_0$ is a *singularity* of F. If there exists a neighborhood around s_0 which contains no other singularities of F, then we say that s_0 is an *isolated singularity* of F.

∎

It is standard convention by authors to assume that ∞ is a singularity. This convention shall receive further consideration below. A zero at ∞, we shall see, should be viewed as a *removable singularity* (cf. definition 2.3.8).

Proposition 2.3.2 (Laurent Series)
Let s_0 be an isolated singularity of F. Given this, there exists an annulus

$$A(s_0, r_1, r_2) \stackrel{\text{def}}{=} \{s \in C \mid r_1 < |s - s_0| < r_2\}$$

about s_0, and sequences $\{a_m\}_{m=0}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, such that

$$F(s) = \sum_{m=0}^{\infty} a_m (s - s_0)^m + \sum_{n=1}^{\infty} b_n \frac{1}{(s - s_0)^n}$$

for all $s \in A(s_0, r_1, r_2)$. Moreover, the convergence is uniform on all compact subsets of the annulus. We shall refer to this representation for F as the *Laurent expansion* for F about s_0. The second summation, which consists only of negative powers of $(s - s_0)$, is called the *principal or singular part* of F at s_0.

∎

Let s_0 be an isolated singularity of F.

Definition 2.3.8 (Removable Singularity)
The point $s = s_0$ is said to be a *removable singularity* of F, if

$$\lim_{s \to s_0} (s - s_0) F(s) = 0$$

along any path in the domain of definition of F [2, pp. 124].

∎

One can show that the above is equivalent to F having no principal part. In such a case, the function can be made analytic at s_0, simply by redefining it at the point.

One can also show that s_0 is removable if and only if $|F(s)|$ is bounded in some annulus about s_0. This condition is due to Riemann.

Definition 2.3.9 (Pole)
If $\lim_{s \to s_0} F(s) = \infty$, along any path in the domain of definition of F, then we say that the point $s = s_0$ is a *pole* of F.

∎

One can show that this is equivalent to F having a principal part with only a finite number of terms.

Definition 2.3.10 (Multiplicity of a Pole)
Let s_0 be a pole of F. We say that it is a *pole with multipliciity* m, if s_0 is a zero of $\frac{1}{F}$ with multiplicity m.

■

Definition 2.3.11 (Improper Function)
We say that F is *improper* if ∞ is a pole of F.

■

Definition 2.3.12 (Rational Function)
We say F is a *rational function*, if it is the ratio of two polynomials; each of finite degree. We say that it is *real-rational*, if the coefficients of the numerator and denominator polynomials are real.

■

Rational functions have a finite number of poles and zeros. They also possess singularities at ∞ which are removable.

Definition 2.3.13 (Meromorphic Function)
F is said to be *meromorphic* in a domain $\mathcal{D}$ if the only singularities within $\mathcal{D}$ are isolated poles. If $\mathcal{D} = C$, then we just say that the function F is *meromorphic.*

■

The functions $F(s) = \frac{1}{\sin(s)}$ and $G(s) = \frac{1}{1-e^{-s}}$ are examples of a meromorphic functions.

A meromorphic function can only have a finite number of poles in any compact set. If a meromorphic function has an infinite number of poles, then they must cluster at ∞.

It can be shown that a meromorphic function F with a finite number of poles is necessarily the ratio of an entire function over a polynomial.

It can also be shown that meromorphic functions are necessarily the ratio of two entire functions.

Definition 2.3.14 (Essential Singularity)
We say that s_0 is an *essential singularity* of F, if the principal part of its Laurent expansion about s_0 has an infinite number of nonzero terms.

■

One can show that this can occur if and only if s_0 is a singularity of F which is neither removable nor a pole.

Singularities at ∞ are treated as follows.

Definition 2.3.15 (Singularities at ∞)
$s = \infty$ is a removable singularity, a pole, or an essential singularity of $F(s)$, if $\zeta = 0$ is a removable singularity, a pole , or an essential singularity of $F(\frac{1}{\zeta})$.

■

The functions e^{-s} and $e^{\frac{1}{s}}$ have essential singularities at ∞ and 0, respectively.

The following proposition gives a remarkable characterization of essential singularities.

Proposition 2.3.3 (Picard's Theorem)
A complex-valued function F in any neighborhood of an essential singularity assumes all values except possibly one.

$\blacksquare$

This proposition is often referred to as *Picard's Theorem.* A weaker version of the theorem, known as the *Casorati-Weierstrass Theorem*, appears in [33, pp. 158].

The above shows that isolated singularities of otherwise single-valued analytic functions must either be removable singularities, poles, or essential essential singularities.

The following proposition characterizes "maxima" of analytic functions on a domain [2, pp. 134], [33, pp. 150], [49, pp. 212, 249, 253-259]. It is known as the *Maximum Modulus Theorem.*

Proposition 2.3.4 (Maximum Modulus Theorem)
Let F be analytic within a domain $\mathcal{D}$. Then $|F(s)|$ achieves its maximum within $\mathcal{D}$ if and only if F is constant. If the domain $\mathcal{D}$ is bounded, then we have

$$|F(s)| \leq \sup_{s \in \partial \mathcal{D}} |F(s)|$$

for all $s \in \mathcal{D}$.

$\blacksquare$

The above does not imply that an analytic function F on a domain will achieve its maximum on the boundary. The following example illustrates this point.

Example 2.3.1 (Unbounded Analytic Function on a Strip)
The function $F(s) = e^{e^s}$ is an entire function. Consider it over the unbounded domain $|Im(s)| \leq \frac{\pi}{2}$. It does not achieve its maximum modulus on the boundary. Its magnitude on the boundary is 1. Within the domain, however, the function grows exponentially along the positive real axis.

The following proposition can be found in [2, pp. 127], [49, pp. 209].

Proposition 2.3.5 (Uniqueness of Analytic Functions)
Let F and G be analytic within some domain $\mathcal{D}$ in the complex plane. Let S be a subset of $\mathcal{D}$. If S has an accumulation point in $\mathcal{D}$ and $F = G$ on S then $F = G$ throughout $\mathcal{D}$.

$\blacksquare$

This proposition implies that the zeros of a non-constant analytic function cannot accumulate within its domain of analyticity. If they do, then the function must be identically zero over the entire domain. This property is due to the fact that the zeros of a single-valued analytic function are necessarily isolated within its domain of analyticity.

2.4 Multi–Valued Complex Functions

In this section we consider multi-valued complex functions of a single complex variable. Such functions, as we shall see, can be viewed as a collection of single-valued functions. Let F denote a complex valued function of a single complex variable s.

In order to precisely define multi-valued functions, we define the concept of a *branch point* as follows. A branch point is a singularity that is associated with multi-valued functions only.

Definition 2.4.1 (Branch Point)
The point $s = s_0$ is a *branch point* of the function F, if when we travel around the point, along a sufficiently small circle, we do not return to the same value; i.e. for $r > 0$, sufficiently small,

$$F(s_0 + r) \neq F(s_0 + re^{j2\pi}).$$

■

Branch points at ∞ can be defined similarly by considering balls around the point at ∞. Such a ball could be precisely defined in terms of the *Riemann sphere* associated with the complex plane. An equivalent approach is provided by the following definition. The definition parallels that given for singularities of single-valued functions at ∞ in definition 2.3.15.

Definition 2.4.2 (Branch Points at ∞)
The point $s = \infty$ is a *branch point* of the function $F(s)$, if $\zeta = 0$ is a branch point of $F(\frac{1}{\zeta})$.

■

Definition 2.4.3 (Multi-Valued Function)
F is a *multi-valued* function if it posseses a branch point in its domain of definition.

■

The function $F(s) = \sqrt{s-1}$, for example, is multi-valued. It posseses a branch point at $s = 1$.

The analytic study of multi-valued functions usually requires that the multi-valued function be expressed in terms of single-valued functions. One way of doing this is to consider the multi-valued function in a restricted region of the extended complex plane. Then, one chooses a value at each point in such a way that the resulting single-valued function is continuous on the restricted domain. This motivates the following definition.

Definition 2.4.4 (Branch of a Multi-Valued Function)
A continuous single-valued function obtained from a multi-valued function is called a *branch* of the multi-valued function.

■

Definition 2.4.5 (Branch Cut)
Typically, to obtain a branch, one must delete some curve in the s-plane. Such a curve is typically referred to as a *branch cut.* The curve is such that if not crossed over, the function remains single-valued. Given this, we have that a branch cut is a line or curve of singular points introduced in defining a branch of a multi-valued function.

■

The branch cuts of a multi-valued function are not unique. A given branch cut, however, uniquely defines a distinct branch of the multi-valued function. Branch points are common to all branch cuts of a multi-valued function. A branch is also uniquely determined once a particular value of the multi-valued function over the restricted domain has been specified. Specifying such a value automatically determines the branch cut. This follows by continuity.

We now address multi-valued functions which shall be implicitly consider in the sequel.

Definition 2.4.6 (Complex Logarithm)
Let $\ln(\cdot)$ denote the real-valued natural logarithm from elementary calculus. The complex logarithmic function is denoted $\ln_{mv}(\cdot)$ and defined by the equation

$$\ln_{mv}(s) \stackrel{\text{def}}{=} \ln|s| + j\theta(s)$$

where $s = |s|e^{j\theta(s)}$ is any extended complex number and $\ln_{mv}(0) \stackrel{\text{def}}{=} -\infty$ and $\ln_{mv}(\infty) \stackrel{\text{def}}{=} \infty$.

∎

To see that this function is multi-valued we consider a circle of radius one centered at the origin of the complex plane. We then note that if the circle is traversed in the conterclockwise direction, then $\ln_{mv}(\cdot)$ does not return to its initial value. We have, for example, $\ln_{mv}(1e^{j0}) = 0 + j0$ and $\ln_{mv}(1e^{j2\pi}) = 0 + j2\pi$. The origin is thus a branch point of the complex logarithmic function. Given this, we see that it is a multi-valued function. One can show that the point $s = \infty$ is also a branch point. To obtain a branch of this function we proceed as follows.

Let

$$S \stackrel{\text{def}}{=} C_e/\{-\infty \leq Re(s) < 0\ ;\ Im(s) = 0\} = \{s \in C_e \mid -\pi < \theta(s) < \pi\ \}$$

denote the set of points in the extended complex plane obtained by deleting the extended negative real-axis. S defines a branch cut for the complex logarithmic function.

Definition 2.4.7 (Principal Angle)
We define the *principal angle function* $\theta_{pv} : S/0 \to (-\pi, \pi)$ as follows

$$\theta_{pv}(s) \stackrel{\text{def}}{=} \begin{cases} \tan^{-1}\frac{\omega}{\sigma} & \sigma > 0; \\ -\tan^{-1}\frac{\sigma}{\omega} + \frac{\pi}{2} & \sigma < 0; \omega > 0; \\ -\tan^{-1}\frac{\sigma}{\omega} - \frac{\pi}{2} & \sigma < 0; \omega < 0 \end{cases}$$

where $s = \sigma + j\omega$.

∎

We note that θ_{pv} is continuous on $S/0$.

Definition 2.4.8 (Principal Logarithm)
Given the above, the *principal branch* of the complex logarithmic function is defined on S as follows

$$\ln_{pv}(s) \stackrel{\text{def}}{=} \ln|s| + j\theta_{pv}(s)$$

for $s \neq 0$, $\ln_{pv}(0) \stackrel{\text{def}}{=} -\infty$, and $\ln_{pv}(\infty) \stackrel{\text{def}}{=} \infty$.

∎

This function is analytic, as well as continuous, on S. The negative real-axis is the associated branch cut. Other branch cuts and branches of the logarithmic function can similarly be defined.

Now we consider another multi-valued function. It is defined as follows.

Definition 2.4.9 (Complex Power Function)
Let c be a fixed real number. Given this, we have

$$(s^c)_{mv} \stackrel{\text{def}}{=} e^{c\ln_{mv}(s)}$$

where s is any complex number.

■

The principal branch of this function is defined by using the principal logarithm $\ln_{pv}(\cdot)$ as follows.

Definition 2.4.10 (Principal Power Function)

$$(s^z)_{pv} \stackrel{\text{def}}{=} e^{z \ln_{pv}(s)}$$

■

Proposition 2.4.1 (Analytic Branches)
Given a multi-valued function, an analytic branch F can always be constructed in any domain which does not contain a branch point.

■

This convention shall be adopted throughout the thesis for all multi-valued functions considered. Given this, functions such as

$$F(s) = \left(\frac{1}{s+1}\right)^{\frac{1}{2}} \left(\frac{s}{s+2}\right)^{\frac{1}{3}} \left(\frac{s-j1}{s+3}\right)^{\frac{1}{4}} \left(\frac{s+j1}{s+4}\right)^{\frac{1}{4}}$$

will be regarded as analytic in the extended open right half plane and continuous everywhere on the extended imaginary axis.

It was seen in the previous section that single-valued functions exhibit polynomial behavior near zeros. This is not the case for single-valued functions which are constructed from multi-valued functions. The function $F(s) = \sqrt{s}$, for example, exhibits irrational behavior near $s = 0$. We thus have the following definition.

Definition 2.4.11 (Algebraic Multiplicity of a Zero)
Let F be a branch of a multi-valued function. Let F be analytic at $s = s_0$. Also, let s_0 be a zero of F. We say that s_0 has *algebraic multiplicity* $m \in R_+$, if there exists a function G such that for all s in a neighborhood of s_0,

$$F(s) = (s - s_0)^m G(s)$$

where $G(s_0) \neq 0$ and G is analytic within the neighborhood. Here the neighborhood is assumed to lie within the domain of definition of the branch F.

■

A non-negative real number m and a function G can always be found for any analytic branch F of a multi-valued function.

2.5 $\mathcal{L}^2$ Function Theory

The function space $\mathcal{L}^2$ is defined as follows.

Definition 2.5.1 (Function Space: $\mathcal{L}^2(R)$)
$\mathcal{L}^2 \stackrel{\text{def}}{=} \mathcal{L}^2(R)$ will denote the space of Lebesgue square integrable complex-valued functions with support on R. $\mathcal{L}^2(R_+)$ and $\mathcal{L}^2(R_-)$ are similarly defined. $\mathcal{L}^2$ is a normed linear space over the field C, when endowed with the following norm

$$||f||_{\mathcal{L}^2} \stackrel{\text{def}}{=} \sqrt{\int_{-\infty}^{\infty} |f(t)|^2 dt}.$$

■

The $\mathcal{L}^2$ functions should be thought of as the set of all finite energy time signals.

The space $\mathcal{L}^2$ is also an *inner product space* when endowed with the following inner product

$$< f, g >_{\mathcal{L}^2} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \overline{f(t)} g(t) dt.$$

Moreover, it is *complete* with respect to this inner product. $\mathcal{L}^2$ is thus a *Hilbert space.*

The following proposition can be found in [35]. It is the basis for classical projection theory.

Proposition 2.5.1 (Classical Projection Theorem)
Let $\mathcal{H}$ denote a *Hilbert space* withe inner product $< \cdot, \cdot >_{\mathcal{H}}$. Let $\mathcal{M}$ denote a closed subspace of $\mathcal{H}$. Given $x \in \mathcal{H}$ there exists a unique element $m_o \in M$ such that

$$\min_{m \in \mathcal{M}} ||x - m||_{\mathcal{H}} = ||x - m_o||_{\mathcal{H}}.$$

Moreover, $x - m_o \in \mathcal{M}^{\perp}$ where

$$\mathcal{M}^{\perp} \stackrel{\text{def}}{=} \{x \in \mathcal{H} \mid \ < x, m >_{\mathcal{H}} = 0 \ \ m \in \mathcal{M}\}.$$

We say that m_o is the *projection of x onto* $\mathcal{M}$ and write

$$m_o = \Pi_{\mathcal{M}} x.$$

Also, $\mathcal{H}$ is the *direct sum* of *calM* and *calM*$^{\perp}$; i.e.

$$\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}.$$

This notation means that each element $x \in \mathcal{H}$ can be written as $x = m_1 + m_2$ where $m_1 \in \mathcal{M}$ and $m_2 \in \mathcal{M}^{\perp}$.

■

From the *classical projection theorem,* it thus follows that $\mathcal{L}^2(R)$ is the *direct sum* of $\mathcal{L}^2(R_+)$ and $\mathcal{L}^2(R_-)$:

$$\mathcal{L}^2(R) = \mathcal{L}^2(R_+) \oplus \mathcal{L}^2(R_-);$$

i.e. given $f \in \mathcal{L}^2(R)$ there exists unique functions $f_+ \in \mathcal{L}^2(R_+)$ and $f_- \in \mathcal{L}^2(R_-)$, such that $f = f_+ + f_-$. This is because $\mathcal{L}^2(R_-)$ is the *orthogonal complement* of $\mathcal{L}^2(R_+)$ in $\mathcal{L}^2(R)$.

Definition 2.5.2 (Function Space: $\mathcal{L}^2(jR)$)
One can also define the space $\mathcal{L}^2(jR)$ as the set of all complex-valued functions which are Fourier/Plancherel transforms of $\mathcal{L}^2(R)$ time functions. $\mathcal{L}^2(jR)$ is a normed linear space over the field C when endowed with the following norm

$$||F||_{\mathcal{L}^2(jR)} \stackrel{\text{def}}{=} \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega}.$$

■

Parseval's theorem tells us that $\mathcal{L}^2(R)$ and $\mathcal{L}^2(jR)$ are isomorphic. More specifically, it shows that

$$\|f\|_{\mathcal{L}^2(R)} = \|F\|_{\mathcal{L}^2(jR)}.$$

Since $\mathcal{L}^2(R)$ and $\mathcal{L}^2(jR)$ are isometrically isomorphic, there is no need to distinguish between the spaces. We shall usually just write $\mathcal{L}^2$. Which space is being considered should be apparent from the context.

The function space $\mathcal{H}^2$ is defined as follows.

Definition 2.5.3 (Function Space: $\mathcal{H}^2$)
$\mathcal{H}^2 \stackrel{\text{def}}{=} \mathcal{H}^2(C_+)$ will denote the Hardy space of complex-valued functions which are analytic in C_+ and uniformly Lebesgue square integrable on lines parallel to the imaginary axis in C_+. $\mathcal{H}^2$ is also a normed linear space over the field C, when endowed with the following norm

$$\|F\|_{\mathcal{H}^2} \stackrel{\text{def}}{=} \sup_{\sigma>0} \sqrt{\frac{1}{2\pi} \int_{\sigma-j\infty}^{\sigma+j\infty} |F(\sigma+j\omega)|^2 d\omega}.$$

■

$\mathcal{H}^2$ consists exactly of those functions which are Laplace/Plancherel transforms of $\mathcal{L}^2(R_+)$ functions [29, pp. 100, Paley-Wiener Theorem]. Consequently, there is no need to distinguish between $\mathcal{H}^2$ and $\mathcal{L}^2(R_+)$. They are isomorphic.

Definition 2.5.4 (Function Space: ${\mathcal{H}^2}^\perp$)
Analogously, ${\mathcal{H}^2}^\perp$ will be used to denote the space of functions which are Laplace transforms of functions in $\mathcal{L}^2(R_-)$.

■

It is a fact [30, pp. 128] that $\mathcal{H}^2$ and ${\mathcal{H}^2}^\perp$ functions can be unitarily extended to have support almost everywhere on the imaginary axis. More precisely, we have the following proposition.

Proposition 2.5.2 (Extension of $\mathcal{H}^2$ Functions onto the Imaginary Axis)
Given $F \in \mathcal{H}^2$, it possesses *non-tangential* limits almost everywhere on the imaginary axis. By taking such limits, one obtains a unique extension $\tilde{F} \in \mathcal{L}^2$ of F onto the imaginary axis. Moreover, this extension is an isometry.

■

A similar proposition applies for functions in ${\mathcal{H}^2}^\perp$. When dealing with $\mathcal{H}^2$ and ${\mathcal{H}^2}^\perp$ functions we shall always assume that we are dealing with the extended functions; i.e. we make no distinction between the function and its extension. Given this, the norms of such functions can be computed from values on the imaginary axis. This gives us the following proposition.

Proposition 2.5.3 (Norm of $\mathcal{H}^2$ Functions)
Given $F \in \mathcal{H}^2$, we have

$$\|F\|_{\mathcal{H}^2} = \|F\|_{\mathcal{L}^2(jR)}.$$

∎

The following notation shall be used to denote the various projection operators on $\mathcal{L}^2$.

Definition 2.5.5 (Projection Operators on $\mathcal{L}^2$)
The projection of $\mathcal{L}^2$ onto $\mathcal{L}^2(R_+)$ (or $\mathcal{H}^2$) shall be denoted $\Pi_{\mathcal{L}^2(R_+)}$ (or $\Pi_{\mathcal{H}^2}$) . These operators shall be used interchangeably. The projection of $\mathcal{L}^2$ onto $\mathcal{L}^2(R_-)$ (or $\mathcal{H}^{2\perp}$) shall be denoted $\Pi_{\mathcal{L}^2(R_-)}$ (or $\Pi_{\mathcal{H}^{2\perp}}$). These operators shall be used interchangeably.

∎

We note that such projection operators have induced norms, on $\mathcal{L}^2$, equal to 1.

2.6 $\mathcal{L}^\infty$ Function Theory

The function space $\mathcal{L}^\infty$ is defined as follows.

Definition 2.6.1 (Function Space: $\mathcal{L}^\infty(jR)$)
$\mathcal{L}^\infty \stackrel{\text{def}}{=} \mathcal{L}^\infty(jR)$ will denote the space of Lebesgue measurable essentially bounded complex-valued functions with support on the imaginary axis. This space is a normed linear space over the field C, when endowed with the following norm

$$\|F\|_{\mathcal{L}^\infty} \stackrel{\text{def}}{=} ess \sup_{\omega \in R_e} |F(j\omega)|.$$

∎

Proposition 2.6.1 (System Interpretation)
Given $F \in \mathcal{L}^\infty$, the associated time function f defines a convolution kernel from $\mathcal{L}^2$ to $\mathcal{L}^2$. Moreover,

$$\|F\|_{\mathcal{L}^\infty} = \sup_{\|x\|_{\mathcal{L}^2} \leq 1} \|f * x\|_{\mathcal{L}^2}$$

where $f * x$ denotes the convolution of f with x.

∎

Definition 2.6.2 (Adjoint)
Given a complex-valued function $F(s)$ with domain of definition including the imaginary axis, we shall denote the *adjoint* of F as follows

$$F^*(s) \stackrel{\text{def}}{=} \overline{F(-\bar{s})}.$$

∎

Strictly speaking, the concept of an adjoint should be defined in terms of bounded linear functionals [48]. This degree of rigor will not be necessary for our purposes.

The function space $C_e(jR)$ is defined as follows.

Definition 2.6.3 (Function Space: $C_e(jR)$)
$C_e \stackrel{\text{def}}{=} C_e(jR)$ is the set of all complex-valued functions which are continuous on the extended imaginary axis.

∎

2.7 $\mathcal{H}^\infty$ Function Theory

The function space $\mathcal{H}^\infty$ is defined as follows.

Definition 2.7.1 (Function Spaces: $\mathcal{H}^\infty, \mathcal{H}_0^\infty$)
$\mathcal{H}^\infty \stackrel{\text{def}}{=} \mathcal{H}^\infty(C_+)$ will denote the *Hardy space* of complex-valued functions which are analytic and essentially bounded in C_+. $\mathcal{H}^\infty$ is a normed linear space over the field C, when endowed with the norm

$$\|F\|_{\mathcal{H}^\infty} \stackrel{\text{def}}{=} \sup_{\sigma>0} \sup_{\omega \in R_e} |F(\sigma + j\omega)|.$$

$\mathcal{H}_0^\infty \stackrel{\text{def}}{=} \mathcal{H}_0^\infty(C_+)$ will denote the subspace of $\mathcal{H}^\infty$ functions which roll-off.

∎

$R\mathcal{H}^\infty$ and $R\mathcal{H}_0^\infty$ will denote the corresponding subspaces of real-rational $\mathcal{H}^\infty$ functions. We note that $R\mathcal{H}^\infty$ functions have no poles in the extended closed right half plane [23]; i.e. all of their poles lie in the open left half plane.

Definition 2.7.2 (Function Space: $\mathcal{H}^\infty(C_-)$)
$\mathcal{H}^\infty(C_-)$ will denote the *Hardy space* of complex functions which are analytic and essentially bounded in C_-.

∎

$R\mathcal{H}^\infty(C_-)$ will denote the corresponding subspace of real-rational functions. Such functions have no poles in the extended closed left half plane [23]; i.e. all of their poles lie in the open right half plane.

Definition 2.7.3 ($\mathcal{H}^B$)
$\mathcal{H}^B \stackrel{\text{def}}{=} \mathcal{H}^B(C_+)$ will be used to denote the *Hardy space* of complex-valued functions which are analytic in C_+ and bounded on compact subsets within C_+ [60, pp. 589].

∎

This space should be thought of as containing all improper functions which would be in $\mathcal{H}^\infty$ if it were not for their improperness (e.g. $f(s) = s$). $R\mathcal{H}^B$ will denote the corresponding subspace of real-rational functions.

It is a fact [30, pp. 128] that $\mathcal{H}^\infty$ functions can be unitarily extended to have support almost everywhere on the imaginary axis. More precisely, we have the following proposition.

Proposition 2.7.1 (Extension of $\mathcal{H}^\infty$ Functions onto the Imaginary Axis)
Given $F \in \mathcal{H}^\infty$, it possesses *non-tangential* limits almost everywhere on the imaginary axis. By taking such limits, one obtains a unique extension $\tilde{F} \in \mathcal{L}^\infty$ of F onto the imaginary axis. Moreover, this extension is an isometry.

∎

When dealing with such functions we shall always assume that we are dealing with the extended functions; i.e. we make no distinction between the function and its extension. Given this, the norms of such functions can be computed from values on the imaginary axis. We state this formally as follows.

Proposition 2.7.2 (Maximum Modulus Theorem for $\mathcal{H}^\infty$)
Given $F \in \mathcal{H}^\infty$, we have

$$\|F\|_{\mathcal{H}^\infty} = \|F\|_{\mathcal{L}^\infty} \stackrel{\text{def}}{=} ess \sup_{\omega \in R_e} |F(j\omega)|.$$

Consequently,

$$|F(s)| \leq \|F\|_{\mathcal{L}^\infty}$$

for all s such that $Re(s) > 0$.

∎

Definition 2.7.4 (Uniform Roll-off)
We shall say that a sequence of functions $\{F_n\}_{n=1}^{\infty} \subset \mathcal{H}_0^\infty$ *uniformly rolls-off* if there exists a frequency $\omega_0 \stackrel{\text{def}}{=} \omega_0(\epsilon)$ such that $|F_n(j\omega)| \leq \epsilon$ for all $n \in Z_+$ for all ω such that $|\omega| \geq \omega_0$.

∎

Definition 2.7.5 (Inner Function)
Given $F \in \mathcal{H}^\infty$, we say that F is *inner* in $\mathcal{H}^\infty$ if $F^*(j\omega)F(j\omega) = 1$ almost everywhere on the extended imaginary axis.

∎

$R\mathcal{H}^\infty$ functions which are inner have all of their poles in the open left half plane and all of their zeros within the open right half plane. They do not have any zeros on the extended imaginary axis.

A special class of inner functions are Blaschke products [30, pp. 63-68; pp. 132]. The following proposition desribes their properties.

Proposition 2.7.3 (Blaschke Product)
Let $\{z_k\}_{k=1}^{\infty}$ denote points in the open right half plane such that $\sum_{k=1}^{\infty} \frac{Re(z_k)}{1+|z_k|^2} < \infty$. The function

$$B(s) \stackrel{\text{def}}{=} \prod_{k=1}^{\infty} \frac{|1-z_k^2|}{1-z_k^2} \frac{s-z_k}{s+\overline{z_k}}$$

is well defined and is called a *Blaschke product.* The *Blaschke condition*

$$\sum_{k=1}^{\infty} \frac{Re(z_k)}{1+|z_k|^2} < \infty$$

is necessary and sufficient for pointwise convergence. Suppose that this condition is satisfied. B then possesses the following properties.

(1) B is inner in $\mathcal{H}^\infty$ and $|B(s)| < 1$ in the open right half plane.

(2) The partial products converge uniformly on compact subsets of the open right half plane.

(3) Let K denote the compact set consisting of the points $-\overline{z_k}$ and the accumulation points of z_k. The convergence of the partial products is uniform on any closed subset of the complex plane

which is disjoint from K. B is thus analytic off K.

(4) B has an essential singularity at each point of accumulation of the z_k. Hence B cannot be extended continuously from the open right half plane to any such accumulation point, for the extended value of B would have to be zero, while the non-tangential limits of B are of modulus 1 almost everywhere.

■

The above proposition implies that infinite Blaschke products are not continuous everywhere on the extended imaginary axis. Hence, by proposition 2.10.1, they cannot be uniformly approximated by $\mathcal{RH}^\infty$ functions. Moreover, $B \in \mathcal{C}_e$ if and only if B is a finite product.

Blaschke products are the only inner functions which possess zeros. We now define the notion of a *singular inner function.*

Definition 2.7.6 (Singular Inner Function)
A *singular inner function* is an inner function which has no zeros.

■

The function $F(s) = e^{-s}$ is inner and it has no zeros. It is a singular inner function.

Proposition 2.7.4 (Properties of Singular Inner Functions)
Let S denote a singular inner function. Then S is uniquely determined by singular positive measure μ on the imaginary axis. S is analytic everywhere in the complex plane except at those points on the imaginary axis which lie in the closed support of the measure μ. The function S cannot be continuously extended from the open right half plane to any point in the closed support of μ.

■

The above proposition implies that singular inner functions are not continuous everywhere on the extended imaginary axis. Hence, by proposition 2.10.1, they cannot be uniformly approximated by $\mathcal{RH}^\infty$ functions. An example of such a function is $F(s) = e^{-s}$.

Proposition 2.7.5 (Factorization of Inner Functions)
Every inner function $F \in \mathcal{H}^\infty$ can be written as the product of a Blaschke product and a singular inner function.

■

Definition 2.7.7 (Outer Function)
Given $F \in \mathcal{H}^\infty$, we say that F is *outer* in $\mathcal{H}^\infty$ if $F\mathcal{H}^2$ is *dense* in $\mathcal{H}^2$ with respect to the topology induced by the norm $\|\cdot\|_{\mathcal{H}^2}$ on $\mathcal{H}^2$; i.e given $y \in \mathcal{H}^2$ there exists $x \in \mathcal{H}^2$ such that $\|Fx - y\|_{\mathcal{H}^2} \leq \epsilon$.

■

The above condition should be thought of, loosely speaking, as a "surjectivity" condition. $\mathcal{RH}^\infty$ functions which are outer have all of their poles in the open left half plane and all of their zeros outside of the open right half plane. Such functions, in general, have zeros on the extended imaginary axis.

Proposition 2.7.6 (Inner-Outer Factorization)
Given $F \in \mathcal{H}^\infty$ there exists inner and outer functions $F_i, F_o \in \mathcal{H}^\infty$ such that

$$F = F_i F_o.$$

∎

This proposition can be found in [49, pp. 344] , [54].

Definition 2.7.8 (Minimum Phase Function)
Given $F \in \mathcal{H}^\infty$, we say that F is *minimum phase* if F has all of its zeros in the open left half plane.

∎

2.8 Theory of Linear Operators

Let $\mathcal{N}_1$ and $\mathcal{N}_2$ denote normed linear spaces over fields $\mathcal{F}_1$ and $\mathcal{F}_2$, with norms $||\cdot||_{\mathcal{N}_1}$ and $||\cdot||_{\mathcal{N}_2}$, respectively. Let $T : \mathcal{N}_1 \to \mathcal{N}_2$ denote a linear operator.

Definition 2.8.1 (Bounded Linear Operator)
We say that T is a *bounded linear operator* from $\mathcal{N}_1$ to $\mathcal{N}_2$ if its *induced norm*, given by

$$||T|| \stackrel{\text{def}}{=} \sup_{||x||_{\mathcal{N}_1}=1} ||Tx||_{\mathcal{N}_2}$$

is finite.

The set of all bounded linear operators from $\mathcal{N}_1$ to $\mathcal{N}_2$ is denoted $\mathcal{B}(\mathcal{N}_1, \mathcal{N}_2)$.

∎

Definition 2.8.2 (Finite Rank Operator)
We say that T is a *finite rank operator* from $\mathcal{N}_1$ to $\mathcal{N}_2$ if the closure of its *range space* $T(\mathcal{N}_1)$, is finite dimensional.

The set of all finite rank operators from $\mathcal{N}_1$ to $\mathcal{N}_2$ is denoted $\mathcal{B}_{oo}(\mathcal{N}_1, \mathcal{N}_2)$.

∎

Definition 2.8.3 (Compact Operator)
We say that T is a *compact operator* from $\mathcal{N}_1$ to $\mathcal{N}_2$ if for any bounded subset $M \subset \mathcal{N}_1$, the closure of $T(M)$ is compact with respect to the topology on $\mathcal{N}_2$ induced by the norm $||\cdot||_{\mathcal{N}_2}$.

The set of all compact operators from $\mathcal{N}_1$ to $\mathcal{N}_2$ is denoted $\mathcal{B}_o(\mathcal{N}_1, \mathcal{N}_2)$.

∎

Finite rank operators are compact. Compact operators, however, are not necessarily finite rank. Their relation to finite rank operators is given by the following proposition.

Proposition 2.8.1 (Finite Rank Approximants)
If $\mathcal{N}$ is either $\mathcal{L}^2$ or $\mathcal{L}^\infty$ then $\mathcal{B}_{oo}(\mathcal{N}, \mathcal{N})$ is dense in $\mathcal{B}_o(\mathcal{N}, \mathcal{N})$ with respect to the induced-norm topology.

∎

This proposition says that compact operators can usually be approximated by finite rank operators. This proposition appears in [5, pp. 179].

2.9 Hankel/Toeplitz Operator Theory

In this section we assume that $F \in \mathcal{L}^\infty$ is given.

Definition 2.9.1 (Hankel Operator)
The *Hankel operator* with symbol, or induced by, F is the map $\Gamma_F : \mathcal{H}^2 \to \mathcal{H}^{2\perp}$ defined by $\Gamma_F g \stackrel{\text{def}}{=} \Pi_{\mathcal{H}^{2\perp}} F g$, where $\Pi_{\mathcal{H}^{2\perp}}$ denotes the projection operator from $\mathcal{L}^2$ onto $\mathcal{H}^{2\perp}$ (or $\mathcal{L}^2(R_-)$). The *operator norm* of Γ_F is given by

$$\|\Gamma_F\| \stackrel{\text{def}}{=} \sup_{\|g\|_{\mathcal{L}^2}=1} \|\Gamma_F g\|_{\mathcal{L}^2}.$$

■

Proposition 2.9.1 (Finite Rank Hankel Operator)
F induces a finite rank Hankel operator if and only if $F \in \mathcal{H}^\infty(C_+) + R\mathcal{H}^\infty(C_-)$. The rank of such an operator is equal to the number of open right half plane poles of F.

■

This proposition is known as Kronecker's Theorem and can be found in [44, pp. 46].

Proposition 2.9.2 (Compact Hankel Operator)
F induces a compact Hankel operator if and only if $F \in \mathcal{H}^\infty(C_+) + \mathcal{C}_e(jR)$.

■

This proposition is known as Hartman's Theorem and can be found in [44, pp. 46]. The following example illustrates the use of the theorem.

Example 2.9.1 (Application of Hartman's Theorem)
Suppose $a, \beta, \Delta \in R$ are given. The Hankel operator induced by

$$F(s) = \frac{1}{as+1}$$

is compact. The Hankel operator induced by

$$G(s) = \frac{s+1}{s+\beta} e^{s\Delta} = e^{s\Delta} + \frac{1-\beta}{s+\beta} e^{s\Delta}$$

is non-compact.

■

Definition 2.9.2 (Toeplitz Operator)
The *Toeplitz operator* with symbol, or induced by, F is the map $\Theta_F : \mathcal{H}^2 \to \mathcal{H}^2$ defined by $\Theta_F g \stackrel{\text{def}}{=} \Pi_{\mathcal{H}^2} F g$, where $\Pi_{\mathcal{H}^2}$ denotes the projection operator from $\mathcal{L}^2$ onto $\mathcal{H}^2$ (or $\mathcal{L}^2(R_+)$). The *operator norm* of Θ_F is given by

$$\|\Theta_F\| \stackrel{\text{def}}{=} \sup_{\|g\|_{\mathcal{L}^2}=1} \|\Theta_F g\|_{\mathcal{L}^2}.$$

■

2.10 $\mathcal{H}^\infty$ Approximation Theory

In this section we present results from $\mathcal{H}^\infty$ approximation theory. These results shall be used throughout the thesis. We begin by presenting various notions of convergence in $\mathcal{H}^\infty$.

Let $\{G_n\}_{n=1}^\infty$ denote a sequence of $\mathcal{H}^\infty$ functions. Also, let G be an element of $\mathcal{H}^\infty$.

Definition 2.10.1 (Uniform Convergence)
We shall say that G_n *converges uniformly* in $\mathcal{H}^\infty$ to G, or that G_n *uniformly approximates* G in $\mathcal{H}^\infty$, if

$$\lim_{n\to\infty} \|G_n - G\|_{\mathcal{H}^\infty} = 0.$$

This defines the notion of *uniform convergence* in $\mathcal{H}^\infty$.

∎

Definition 2.10.2 (Compact Convergence)
We shall say that the sequence $\{G_n\}_{n=1}^\infty$ *converges uniformly on all compact frequency intervals* to G, if

$$\lim_{n\to\infty} \left\|(G_n - G)X_{[-\Omega,\Omega]}\right\|_{\mathcal{H}^\infty} = 0$$

for each $\Omega \in R_+$, however large. This type of convergence shall be referred to as *compact convergence* in $\mathcal{H}^\infty$.

∎

In controlling infinite dimensional systems, an often encountered $\mathcal{H}^\infty$ function is the "delay": $G(s) = e^{-s\Delta}$ where $\Delta > 0$. The following example provides us with one method of generating compact approximants for delays.

Example 2.10.1 (Compact Approximant for a Delay)
Let

$$G_n = \left(\frac{n}{s\Delta + n}\right)^n$$

define a sequence of $R\mathcal{H}^\infty$ functions. It can be shown that the sequence defined by G_n uniformly approximates the $\mathcal{H}^\infty$ function $G = e^{-s\Delta}$ on compact frequency intervals [2, pp. 178].

∎

The following example shows how Pade' approximants can be used to generate compact approximants for delays.

Example 2.10.2 (Pade' Approximant)
The $[n,n]$ Pade' approximants for $P(s) = e^{-s\Delta}$ are given by

$$P_n(s) = \frac{N_{p_n}(s)}{D_{p_n}(s)}$$

where

$$D_{p_n}(s) = \sum_{k=0}^{n} \frac{(2n-k)!\, n!}{2n!\, k!\, (n-k)!}(s\Delta)^k$$

and

$$N_{p_n}(s) = D_{p_n}(-s).$$

We note that the P_n are stable for all n. We also note that they are inner, and hence uniformly bounded. Moreover, they uniformly approximate P on compact frequency intervals. The latter follows from [26] where it is shown that

$$|e^{-j\omega\Delta} - P_n(j\omega)| \leq \begin{cases} 2(\frac{\omega\Delta}{2n}(\frac{e}{\sqrt{2}})^{\frac{1}{2}})^{2n+1} & |\omega\Delta| \leq 2(\frac{\sqrt{2}}{e})^{\frac{1}{2}}n; \\ 2 & \text{elsewhere.} \end{cases}$$

∎

The following example provides us with still another method of generating compact approximants for delays.

Example 2.10.3 (Compact Approximant for a Delay)
If $P(s) = e^{-s\Delta}$, then the sequence defined by

$$P_n(s) = \left(\frac{2n - s\Delta}{2n + s\Delta}\right)^n$$

can be shown to uniformly approximate P on compact frequency intervals. This follows from analysis of the expression

$$|P_n(j\omega) - P(j\omega)| = 2|\sin(\frac{\omega\Delta - 2n\tan^{-1}(\frac{\omega\Delta}{2n})}{2})|$$

and the fact that $\lim_{n\to\infty} 2n\tan^{-1}(\frac{\omega\Delta}{2n}) = \omega\Delta$.

∎

In the sequel, we shall quite often deal with uniform convergence on different portions of the imaginary axis. To deal with this type of convergence, we will need the notion of a *δ-neighborhood.*

Definition 2.10.3 (δ-Neighborhood)
Let ω_0 denote any finite real number. Think of ω_0 as representing a frequency point on the imaginary axis. By a δ-neighborhood of ω_0, denoted $B(\omega_0, \delta)$, we shall mean the collection of points in the following bounded open frequency interval

$$B(\omega_0, \delta) \stackrel{\text{def}}{=} (-\omega_0 - \delta, -\omega_0 + \delta) \cup (\omega_0 - \delta, \omega_0 + \delta)$$

where $\delta > 0$, however small.

The symbol $B(\infty, \delta)$ will denote a δ-neighborhood of ∞. It is defined as the collection of points in the following unbounded open frequency interval

$$B(\infty, \delta) \stackrel{\text{def}}{=} (-\infty, -\frac{1}{\delta}) \cup (\frac{1}{\delta}, \infty).$$

By symmetry $B(-\infty, \delta)$ is the same set.

■

With the above convention, a smaller δ implies a "smaller neighborhood"(i.e. less points). A larger δ implies a "larger neighborhood"(i.e. more points). Given this, it is reasonable to refer to δ as the *size* of the neighborhood $B(\cdot, \delta)$ where $(\cdot)$ is any positive extended real number.

We now define other notions of convergence in $\mathcal{H}^\infty$.

Let $\{\omega_k\}_{k=1}^{l}$ denote an increasing sequence of distinct points on the extended non-negative real axis. Here, $l \in Z_+ \cup \{\infty\}$; i.e. the sequence can be finite or countably infinite.

Definition 2.10.4 (Convergence on Closed Sets)
We shall say that the sequence $\{G_n\}_{n=1}^{\infty}$ converges uniformly to G *everywhere except on open neighborhoods of the points* $\{\omega_k\}_{k=1}^{l}$, if

$$\lim_{n\to\infty} \left\| (G_n - G) X_{R/\cup_{k=1}^{l} B(\omega_k,\delta)} \right\|_{\mathcal{H}^\infty} = 0$$

for each $\delta > 0$, however small. Since the set $R/\cup_{k=1}^{l} B(\omega_k, \delta)$ is always closed, one can equivalently say that the sequence $\{G_n\}_{n=1}^{\infty}$ converges uniformly to G *on all closed frequency intervals excluding the points* $\{\omega_k\}_{k=1}^{l}$.

■

We note that the set $R/\cup_{k=1}^{l} B(\omega_k, \delta)$ need not be bounded; e.g. the set $R/B(0,\delta) = (-\infty, \delta] \cup [\delta, \infty)$ is not bounded. This unboundedness in $R/\cup_{k=1}^{l} B(\omega_k, \delta)$ occurs if and only if ∞ is not amongst the points $\{\omega_k\}_{k=1}^{l}$. When ∞ is amongst these points, the set $R/\cup_{k=1}^{l} B(\omega_k, \delta)$ will be bounded and hence compact. This motivates the following extension of *compact convergence* in $\mathcal{H}^\infty$.

Definition 2.10.5 (Convergence on Compact Sets)
We shall say that the sequence $\{G_n\}_{n=1}^{\infty}$ converges uniformly to G *on all compact frequency intervals excluding the points* $\{\omega_k\}_{k=1}^{l}$ (and necessarily $\pm\infty$), if

$$\lim_{n\to\infty} \left\| (G_n - G) X_{R/B(\infty,\delta)\cup\cup_{k=1}^{l} B(\omega_k,\delta)} \right\|_{\mathcal{H}^\infty} = 0$$

for each $\delta > 0$, however small.

■

Example 2.10.4 (Convergence of Sequences)

(a) The sequence defined by the function $f_m = \frac{s}{s+\frac{1}{m}}$ converges uniformly to unity on all closed frequency intervals excluding the point 0; i.e.

$$\lim_{m\to\infty} \left\| (1 - f_m) X_{R/B(0,\delta)} \right\|_{\mathcal{H}^\infty} = \lim_{m\to\infty} \left\| (1 - f_m) X_{(-\infty,-\delta]\cup[\delta,\infty)} \right\|_{\mathcal{H}^\infty} = 0$$

for each $\delta > 0$, however small.

(b) The sequence defined by the function $g_n = \frac{n}{s+n}$ converges uniformly to unity on all compact frequency intervals; i.e.

$$\lim_{n\to\infty} \left\| (1 - g_n) X_{[-\Omega,\Omega]} \right\|_{\mathcal{H}^\infty} = 0$$

for each $\Omega \in R_+$, however large.

■

In later sections we will deal with convergence in $\mathcal{H}^\infty$. Often we will need to turn uniform convergence in $\mathcal{H}^\infty$ on compact sets into uniform convergence. The following lemma tells us how this can be done.

Lemma 2.10.1 (Convergence in $\mathcal{H}^\infty$)
Let $F \in \mathcal{H}^\infty \cap \mathcal{C}_e$. Also, let $F(j\omega_k) = 0$ for each $k = 1, \ldots, l$. Suppose that $\{G_n\}_{n=1}^\infty$ is a uniformly bounded sequence of elements in $\mathcal{H}^\infty$. Also suppose that $G \in \mathcal{H}^\infty$. If the sequence $\{G_n\}_{n=1}^\infty$ converges uniformly to G everywhere except on open neighborhoods of the points $\{\omega_k\}_{k=1}^l$; i.e. if

$$\lim_{n\to\infty} \left\| (G_n - G) X_{R/\cup_{k=1}^l B(\omega_k,\delta)} \right\|_{\mathcal{H}^\infty} = 0$$

for each $\delta > 0$, however small, then

$$\lim_{n\to\infty} \|(G_n - G)F\|_{\mathcal{H}^\infty} = 0.$$

If instead, we have

$$\lim_{n\to\infty} \left\| (G_n - G) X_{[-\Omega,\Omega]} \right\|_{\mathcal{H}^\infty} = 0$$

for each $\Omega \in R_+$, however large, then

$$\lim_{n\to\infty} \|(G_n - G)F\|_{\mathcal{H}^\infty} = 0$$

for each $F \in \mathcal{H}_0^\infty$.

■

Proof The proof of this lemma follows intuitively using arguments from elementary analysis. We shall prove the first statement for $l = \infty$. The case where l is finite will then follow. The proof of the second result will be similar.

(A) Proof of first result.
To prove the first result we will consider the quantity $|(G_n - G)F|$ on two sets. We shall denote the sets S_δ and R/S_δ. The proof will proceed in three steps. (1) First we show that there exists a set S_δ on which F is small. We then make the quantity $|(G_n - G)F|$ small on S_δ by exploiting the boundedness of G_n and G. (2) We then exploit the boundedness of F on R/S_δ to make $|(G_n - G)F|$ small on R/S_δ by taking n sufficiently large. (3) Finally, we combine the results of (1) and (2).

Step 1: Analysis on S_δ.
Let $n \in Z_+$ be fixed. Since G_n and G are uniformly bounded, there exists $L \in R_+$ such that

$$|(G_n - G)F| \leq L\,|F|$$

almost everywhere on the extended imaginary axis. This inequality shall be used twice below. Since $l = \infty$ and the sequence $\{\omega_k\}_{k=1}^l \subset R_e$ is monotone, it possess an accumulation point, say $A \in R_e$; i.e. A can be a finite non-negative value or ∞ [3].

[3] If $l = \infty$ then it should be noted that the domain of analyticity of F cannot be extended to include the point at which the ω_k accumulate. This follows from proposition 2.3.5. A non-constant analytic function with such accumulation of zeros would necessarily be zero throughout its domain.

Since F is continuous everywhere on the extended imaginary axis, it is continuous at A. Since $F(j\omega_k) = 0$, this implies that

$$F(jA) \stackrel{\text{def}}{=} F(\lim_{k\to\infty} j\omega_k) = \lim_{k\to\infty} F(j\omega_k) = 0.$$

By continuity at A, there exists $\delta_A \stackrel{\text{def}}{=} \delta_A(\epsilon, L) > 0$ such that

$$\left\| F X_{B(A,\delta_A)} \right\|_{\mathcal{H}^\infty} \leq \frac{\epsilon}{L}.$$

Combining this with the previous inequality, then gives the following result

$$\left\| (G_n - G) F X_{B(A,\delta_A)} \right\|_{\mathcal{H}^\infty} \leq \epsilon.$$

Since A is an accumulation point of the sequence $\{\omega_k\}_{k=1}^{l}$, the set $B(A, \delta_A)$ contains all but a finite number of the ω_k. Suppose the points $\{\omega_k\}_{k=1}^{l_1}$ lie outside $B(A, \delta_A)$.

Since F is continuous at each ω_k and since $F(j\omega_k) = 0$, then for each $k = 1, 2, \ldots, l_1$, there exists $\delta_k \stackrel{\text{def}}{=} \delta_k(\epsilon, \omega_k, L) > 0$ such that

$$\left\| F X_{\cup_{k=1}^{l_1} B(\omega_k,\delta_k)} \right\|_{\mathcal{H}^\infty} \leq \frac{\epsilon}{L}.$$

From this, and the first inequality $|G_n - G| \leq L\ |F|$, we obtain the following result

$$\left\| (G_n - G) F X_{\cup_{k=1}^{l_1} B(\omega_k,\delta_k)} \right\|_{\mathcal{H}^\infty} \leq \epsilon.$$

We now define

$$\delta \stackrel{\text{def}}{=} \delta(\epsilon, A, L, \{\omega_k\}_{k=1}^{l_1}) \stackrel{\text{def}}{=} \min\{\ \delta_A,\ \min_{k=1,\ldots l_1} \delta_k\ \}.$$

Note that this minimum is well defined and strictly positive since we are minimizing over a finite number of objects. Given this, we can define the set S_δ as follows

$$S_\delta \stackrel{\text{def}}{=} B(A, \delta) \cup \cup_{k=1}^{l_1} B(\omega_k, \delta).$$

With this definition, combining the above results gives the following

$$\|(G_n - G) F X_{S_\delta}\|_{\mathcal{H}^\infty} \leq \epsilon.$$

Moreover, this holds for all $n \in Z_+$. This concludes the analysis on S_δ. We now turn to R/S_δ.

Step 2: Analysis on R/S_δ.

Since F is bounded, there exists $M \in R_+$ such that

$$|(G_n - G)F| \leq M\ |G_n - G|$$

almost everywhere on the extended imaginary axis. Since G_n converges to G uniformly everywhere except on open neighborhoods of the points $\{\omega_k\}_{k=1}^{l}$, we have

$$\lim_{n\to\infty} \left\| (G_n - G) X_{R/S_\delta} \right\|_{\mathcal{H}^\infty} = 0.$$

This implies that there exists $N \stackrel{\text{def}}{=} N(\epsilon, \delta, M)$ such that

$$\left\|(G_n - G)FX_{R/S_\delta}\right\|_{\mathcal{H}^\infty} \le \epsilon$$

for all $n \ge N$. This completes the analysis on R/S_δ.

Step 3: Combine steps 1 and 2.
Combining the results of steps 1 and 2 gives

$$\|(G_n - G)F\|_{\mathcal{H}^\infty} \le \epsilon$$

for all $n \ge N$.

This proves the first result for $l = \infty$. When l is finite the sequence $\{\omega_k\}_{k=1}^l$ has no accumulation point and so the proof is simplified.

(B) Proof of second result.

The proof of the second result is similar to that of the first. Since G_n and G are uniformly bounded we have $|(G_n - G)F| \le L\,|F|$ almost everywhere on the extended imaginary axis. Since $F \in \mathcal{H}_0^\infty$, there exists $\Omega \stackrel{\text{def}}{=} \Omega(\epsilon, L) > 0$ such that

$$\left\|(G_n - G)FX_{B(\infty,\frac{1}{\Omega})}\right\|_{\mathcal{H}^\infty} \le \epsilon$$

for any $n \in Z_+$. By assumption, $\lim_{n\to\infty} \left\|(G_n - G)X_{[-\Omega,\Omega]}\right\|_{\mathcal{H}^\infty} = \lim_{n\to\infty} \left\|(G_n - G)X_{R/B(\infty,\frac{1}{\Omega})}\right\|_{\mathcal{H}^\infty} = 0$. Since F is bounded we have $|(G_n - G)F| \le M\,|G_n - G|$ almost everywhere on the extended imaginary axis. This then implies that

$$\lim_{n\to\infty} \left\|(G_n - G)FX_{R/B(\infty,\frac{1}{\Omega})}\right\|_{\mathcal{H}^\infty} = 0.$$

Combining the above gives

$$\lim_{n\to\infty} \|(G_n - G)F\|_{\mathcal{H}^\infty} = 0.$$

This proves the second result and completes the proof. ∎

This lemma will be applied often in sections to come. The key to the lemma is that F, loosely speaking, "rolls-off" on portions of the imaginary axis where the convergence of G_n to G may be poor. In light of this, we see that roll-off is desirable in certain situations. We thus define the following special *roll-off* functions.

Definition 2.10.6 (Roll-off Functions)

$$f_m(s) \stackrel{\text{def}}{=} \left(\frac{a}{s+a}\right)^{\frac{1}{m}}$$

$$g_n(s) \stackrel{\text{def}}{=} \left(\frac{s^2 + \omega_k^2}{s^2 + 2bs + \omega_k^2 + b^2}\right)^{\frac{1}{n}}$$

$$h_q(s) \stackrel{\text{def}}{=} \left(\frac{q}{s+q}\right)^{l_1} \prod_{k=1}^{l_2} \left(\frac{s^2 + \omega_k^2}{s^2 + 2\frac{1}{q}s + \omega_k^2 + \frac{1}{q}^2}\right)^{m_k}$$

where $a, b \in R_+$, $l_1, l_2, m_k \in Z_+$ are fixed, and $m, n, q \in Z_+$.

■

These functions shall be instrumental in the work that follows. It is important to note several properties about the functions f_m, g_n, and h_q.

First note that they each belong to $\mathcal{H}^\infty$ and have magnitudes, and hence norms, which are less than or equal to unity. Moreover, they are continuous everywhere on the extended imaginary axis. Secondly, one should note that they each approximate unity as follows.

The sequence $\{f_m\}_{m=1}^{\infty}$ uniformly converges to unity on all compact frequency intervals; i.e.

$$\lim_{m\to\infty}\left\|(1-f_m)X_{[-\Omega,\Omega]}\right\|_{\mathcal{H}^\infty}=0$$

for each $\Omega\in R_+$, however large. The sequence $\{g_n\}_{n=1}^{\infty}$ uniformly converges to unity everywhere except on open neighborhoods of the single point ω_k; i.e.

$$\lim_{n\to\infty}\left\|(1-g_n)X_{R/B(\omega_k,\delta)}\right\|_{\mathcal{H}^\infty}=0$$

for each $\delta>0$, however small. The sequence $\{h_q\}_{q=1}^{\infty}$ uniformly converges to unity on all compact frequency intervals excluding the points $\{\omega_k\}_{k=1}^{l_2}$ (and ∞); i.e.

$$\lim_{q\to\infty}\left\|(1-h_q)X_{R/B(\infty,\delta)\cup\cup_{k=1}^{l_2}B(\omega_k,\delta)}\right\|_{\mathcal{H}^\infty}=0$$

for each $\delta>0$, however small.

The irrational functions f_m and g_n have appeared in [14]-[16], [53]. They possess the special property that their phase can be made arbitrarily small, uniformly in frequency, by taking m and n, respectively, sufficiently large; i.e.

$$\lim_{m\to\infty}\|\theta_{f_m}\|_{\mathcal{L}^\infty}=0$$

and

$$\lim_{n\to\infty}\|\theta_{g_n}\|_{\mathcal{L}^\infty}=0.$$

We note, for example, that

$$|\theta_{f_m}|_{[s=j\omega]}=|\frac{-\arctan\frac{\omega}{a}}{m}|\le\frac{\pi}{2m}.$$

This special phase property of f_m and g_n shall be very important in what follows.

The rational function h_q has appeared in [56], [59],[60]. It does not possess the special phase properties that f_m and g_n possess.

The following lemma is analogous to lemma 2.10.1. It will allow us to turn compact convergence in $\mathcal{H}^\infty$ into uniform convergence in $\mathcal{H}^2$.

Lemma 2.10.2 (Convergence in $\mathcal{H}^2$)
Suppose that G_n, $G\in\mathcal{H}^\infty$, where the G_n are uniformly bounded in $\mathcal{H}^\infty$. If

$$\lim_{n\to\infty}\left\|(G_n-G)X_{[-\Omega,\Omega]}\right\|_{\mathcal{H}^\infty}=0$$

for each $\Omega\in R_+$, then

$$\lim_{n\to\infty}\|(G_n-G)F\|_{\mathcal{H}^2}=0$$

for each $F\in\mathcal{H}^2$.

■

Proof The proof of this follows using the natural roll-off of $\mathcal{H}^2$ functions and arguments similar to those used in the proof of lemma 2.10.1. We have

$$\|(G_n - G)F\|^2_{\mathcal{H}^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |(G_n - G)F|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{|\omega|\leq\Omega} |(G_n - G)F|^2 d\omega + \frac{1}{2\pi}\int_{|\omega|\geq\Omega} |(G_n - G)F|^2 d\omega$$

$$\leq \left\|(G_n - G)X_{[-\Omega,\Omega]}\right\|^2_{\mathcal{H}^\infty} \frac{1}{2\pi}\int_{|\omega|\leq\Omega} |F|^2 d\omega + \|G_n - G\|^2_{\mathcal{H}^\infty} \frac{1}{2\pi}\int_{|\omega|\geq\Omega} |F|^2 d\omega.$$

Since G_n and G are uniformly bounded in $\mathcal{H}^\infty$, there exists $M \in R_+$ such that

$$\|(G_n - G)F\|^2_{\mathcal{H}^2} \leq \left\|(G_n - G)X_{[-\Omega,\Omega]}\right\|^2_{\mathcal{H}^\infty} \|F\|^2_{\mathcal{H}^2} + M\int_{|\omega|\geq\Omega} |F|^2 d\omega.$$

Since $F \in \mathcal{H}^2$, the second term can be made arbitrarily small by taking Ω sufficiently large; i.e. there exists $\Omega_o \stackrel{\text{def}}{=} \Omega_o(\epsilon, M, F) \in R_+$ such that

$$\|(G_n - G)F\|^2_{\mathcal{H}^2} \leq \left\|(G_n - G)X_{[-\Omega_o,\Omega_o]}\right\|^2_{\mathcal{H}^\infty} \|F\|^2_{\mathcal{H}^2} + \epsilon$$

for all $n \in Z_+$. Given this, and the assumption, the first term can be made arbitrarily small by taking n sufficiently large; i.e. there exists $N \stackrel{\text{def}}{=} N(\epsilon, \Omega_o, F) \in Z_+$ such that

$$\|(G_n - G)F\|^2_{\mathcal{H}^2} \leq 2\epsilon$$

for all $n \geq N$. This proves the result. ■

In the sequel we will require the following proposition on the uniform approximation of $\mathcal{H}^\infty$ functions by $R\mathcal{H}^\infty$ functions. It tells us, for example, that the irrational function f_m can be approximated uniformly in $\mathcal{H}^\infty$ by $R\mathcal{H}^\infty$ functions.

Proposition 2.10.1 (Existence of $R\mathcal{H}^\infty$ Approximants)
A function $F \in \mathcal{H}^\infty$ can be uniformly approximated by $R\mathcal{H}^\infty$ functions if and only if $F \in \mathcal{C}_e$. Suppose $F \in \mathcal{C}_e$. In such a case, the approximants can be constructed such that they roll-off if and only if F rolls-off.

If in addition, F is minimum phase, then the approximants can be constructed such that they are minimum phase[4].

The approximants can be constructed such that they are inner if and only if F is inner.

Finally, suppose F is also outer with a finite number of zeros on the extended imaginary axis; each with finite algebraic multiplicity. In such a case the approximants can be constructed such that they are minimum phase with the exception of zeros of multiplicity 1 on the imaginary axis at the imaginary zeros of F.

■

Proof

The key elements of this proposition can be found in [28] and [30, pp. 17 - 18]. ■
Related to this proposition are the theorems of Runge, Mittag-Leffler, and Mergelyan [49].

The following example illustrates how one might approximate an outer function with minimum phase functions.

[4]Minimum phase functions have all of their zeros in the open left half plane.

Example 2.10.5 (Approximation of an Outer Function)

Let $F(s) = \sqrt{\frac{s}{s+1}}$. We note that $F \in \mathcal{H}^\infty \cap \mathcal{C}_e$ and is outer. We now show how F can be uniformly approximated by $\mathcal{RH}^\infty$ functions which are minimum phase with the exception of a zero at $s = 0$ with multiplicity 1.

The sequence defined by $F_n \stackrel{\text{def}}{=} s\, G_n$, where $G_n \stackrel{\text{def}}{=} \frac{1}{\sqrt{s+\frac{1}{n}}} \; \frac{1}{\sqrt{s+1}}$ approximates F uniformly in $\mathcal{H}^\infty$. The function G_n is minimum phase, and can be approximated uniformly by minimum phase $\mathcal{RH}^\infty$ functions $\tilde{G}_n$. Given this, the sequence defined by $\tilde{F}_n \stackrel{\text{def}}{=} s\, \tilde{G}_n$ also uniformly approximates F.

∎

The following proposition can be found in [2, pp. 176-177].

Proposition 2.10.2 (Compact Convergence and Analyticity)
Let $\mathcal{D}$ be a domain in the complex plane. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of analytic functions on $\mathcal{D}$ which converge uniformly to F on all compact subsets of $\mathcal{D}$. It then follows that F is analytic on $\mathcal{D}$. Moreover, the derivatives of F_n (of any order) converge to those of F uniformly on all compact subsets of $\mathcal{D}$.

∎

2.11 Duality Theory

Let $\mathcal{N}$ be a normed linear space, over a field $\mathcal{F}$, with norm $||\cdot||_{\mathcal{N}}$. Duality theory deals with the study of bounded linear functionals.

Definition 2.11.1 (Bounded Linear Functional)
Let $f : \mathcal{N} \to \mathcal{F}$ be a linear map. Such a map is called a *linear functional* on $\mathcal{N}$. We say that f is a *bounded linear functional* on $\mathcal{N}$, if its induced norm

$$||f|| \stackrel{\text{def}}{=} \sup_{||x||_{\mathcal{N}}=1} |f(x)|$$

is finite.

∎

Definition 2.11.2 (Dual Space)
The set of all bounded linear functionals on $\mathcal{N}$ is denoted $\mathcal{N}^*$. This set defines a normed linear space over the field $\mathcal{F}$, when endowed with the induced-norm. It is called the *dual space* of $\mathcal{N}$. The space $\mathcal{N}$ is often called the *primal space.*

∎

Definition 2.11.3 (Weak* Convergence)
Let $\{f_n\}_{n=1}^{\infty}, f \subset \mathcal{N}^*$. We say that f_n converges to f in the *weak* topology on* $\mathcal{N}^*$, if

$$\lim_{n\to\infty} |f_n(g) - f(g)| = 0$$

for all $g \in \mathcal{N}$. In such a case we say that f is the *weak* limit* of f_n.

■

The above shows that weak* convergence is nothing more than pointwise convergence on the *primal space* $\mathcal{N}$.

Proposition 2.11.1 (Implication of Weak* Convergence)
Suppose $\{f_n\}_{n=1}^{\infty}, f \subset \mathcal{N}^*$ and that f_n is weak* convergent to f. It then follows that

$$\|f\|_{\mathcal{N}^*} \leq \lim_{n\to\infty} \inf_{k\geq n} \|f_k\|_{\mathcal{N}^*}.$$

■

A proof of this can be found in [5]. It is based on the *Uniform Boundedness (Banach-Steinhauss) Theorem.* This "deep" result is extremely powerful. It shall be exploited in subsequent chapters.

For normed linear spaces the following proposition provides a complete characterization of compactness.

Proposition 2.11.2 (Compact Set)
A subset of $\mathcal{N}$ is *compact* if and only if every sequence in $\mathcal{N}$ has a subsequence which converges to a point in $\mathcal{N}$. In such a case one says the the subset possesses the *Bolzano-Weierstrass* property.

■

The above is not true for general topological spaces. Some authors use the phrase *sequential compactness* to characterize the above behavior. Since we shall only be working with normed linear spaces, such preciseness is not required.

Whether a normed linear space is finite dimensional or infinite dimensional can be determined by ascertaining whether or not its closed unit ball is compact. This is seen from the following proposition.

Proposition 2.11.3 (Finite Dimensionality)
$\mathcal{N}$ is finite dimensional if and only if its closed unit ball is compact.

■

This proposition shows that infinite dimensional spaces necessarily have non-compact closed unit balls.

The following proposition gives us a very useful topological property of the closed unit ball in the dual space $\mathcal{N}^*$.

Proposition 2.11.4 (Alaoglu's Theorem)
The closed unit ball in $\mathcal{N}^*$ is weak* compact; i.e every subsequence has a weak* convergent subsequence.

■

This proposition can often be used to prove the existence of certain limits. Its proof can be found in [5, pp. 134].

The following proposition is an application of Alaoglu's theorem.

Proposition 2.11.5 (On Weak* Convergence)
Suppose $S \subset \mathcal{N}^*$ is weak* compact and $\{f_n\}_{n=1}^{\infty} \subset S$. Suppose also that every weak* convergent subsequence of $\{f_n\}_{n=1}^{\infty}$ converges to $L \in S$. Then, f_n has L as its weak* limit.

■

Proof The proof of this proposition follows readily using a contradiction argument.

Suppose that L is not the weak* limit of f_n. If this is the case then there exists a subsequence $\{f_{n(k)}\}_{k=1}^{\infty}$ and an element $g \in \mathcal{N}$ such that

$$|f_{n(k)}(g) - L(g)| \geq \epsilon$$

for all k.

The subsequence $\{f_{n(k)}\}_{k=1}^{\infty}$ is contained in the compact set S. It is thus uniformly bounded in $\mathcal{N}^*$. By Alaoglu's theorem it follows that it possesses a weak* convergent subsequence. By assumption this subsequence must converge to L. This contradicts the above inequality. It must therefore be that L is, in fact, the weak* limit of the sequence $\{f_n\}_{n=1}^{\infty}$. This completes the proof. ■

An analogous proposition can be proved for other topologies and notions of convergence. The idea is intuitive: given compactness and a unique limit point guarantees convergence to the limit point.

2.12 Equicontinuity, Normality, Arzela-Ascoli

In this section we shall follow the developement given in [49, pp. 245, 281-282]. Let $\mathcal{F}$ denote a family of complex-valued functions defined on a domain $\mathcal{D}$ in the complex plane.

Definition 2.12.1 (Equicontinuous Family)
We say that $\mathcal{F}$ is *equicontinuous* on $\mathcal{D}$, if given $\epsilon > 0$, there exists $\delta \stackrel{\text{def}}{=} \delta(\epsilon) > 0$, such that

$$\sup_{f \in \mathcal{F}} |f(s_1) - f(s_2)| \leq \epsilon$$

for all $s_1, s_2 \in \mathcal{D}$ satisfying $|s_1 - s_2| \leq \delta$.

■

We note that the functions of an equicontinuous family are necessarily uniformly continuous.

Definition 2.12.2 (Pointwise Bounded Family)
We say that $\mathcal{F}$ is *pointwise bounded* on $\mathcal{D}$, if given $s \in \mathcal{D}$, there exist $M(s) < \infty$, such that

$$\sup_{f \in \mathcal{F}} |f(s)| \leq M(s).$$

■

Definition 2.12.3 (Uniformly Bounded Family on Compact Sets)
We say that $\mathcal{F}$ is *uniformly bounded on compact subsets* of $\mathcal{D}$, if given a compact subset $S \subset \mathcal{D}$, there exists $M(S) < \infty$, such that

$$\sup_{\{f \in \mathcal{D};\, s \in S\}} |f(s)| \leq M(s).$$

■

Definition 2.12.4 (Normal Family)
We say that $\mathcal{F}$ is *normal* on $\mathcal{D}$, if every sequence of functions in $\mathcal{F}$ contains a subsequence which converges uniformly on all compact subsets of $\mathcal{D}$. In such a case we say that every sequence in $\mathcal{F}$ exhibits the Bolzano-Weierstrauss property on compact subsets of $\mathcal{D}$.

■

The following specialized version of the *Arzela-Ascoli theorem* gives sufficient conditions for normality.

Proposition 2.12.1 (Arzela-Ascoli Theorem)
Let $\mathcal{F}$ be a family of complex-valued analytic functions defined on a domain $\mathcal{D}$ in the complex plane. If $\mathcal{F}$ is uniformly bounded on each compact subset of $\mathcal{D}$, then $\mathcal{F}$ is normal on $\mathcal{D}$.

■

The following example illustrates how the Arzela-Ascoli Theorem may be applied. Arguments similar to those used in the example shall play a critical role in the later chapters.

Example 2.12.1 (Application)
Let $\{Z_n\}_{n=1}^{\infty} \subset \mathcal{H}^{\infty}$ be a uniformly bounded sequence. The above proposition implies that $\{Z_n\}_{n=1}^{\infty}$ is a normal family. Consequently, there exists a subsequence $\{Z_{n(k)}\}_{k=1}^{\infty}$ which converges uniformly, to say L, on compact subsets in the open right half plane. Since $Z_{n(k)}$ is uniformly bounded in $\mathcal{H}^{\infty}$, L will be bounded in the open right half plane. From proposition 2.10.2, it then follows that $L \in \mathcal{H}^{\infty}$.

2.13 Summary

In this chapter we established notation to be used throughout the thesis. The spaces $\mathcal{H}^{\infty}$ and $\mathcal{H}^2$ were defined and discussed. Results from $\mathcal{H}^{\infty}$ approximation theory were also presented.

Chapter 3

Results from Algebraic Systems Theory

3.1 Introduction

In this chapter we present several results from algebraic systems theory.

3.2 The Youla et al. Parameterization

Throughout the thesis, we shall focus on feedback systems having the classical structure shown in Figure 3.1.

Here F denotes a *plant* and K denotes a *compensator*. We shall assume that both are causal, linear time invariant, single-input single-output, continuous time systems. At the moment, we make no assumption about their dimensions; i.e. each can be finite dimensional or infinite dimensional systems. r and d denote *exogenous signals*. e and u denote the *error* and *control signals*, respectively.

Let $H(F,K)$ denote the transfer function matrix from r,d to e,u. We then have

$$\begin{bmatrix} e \\ u \end{bmatrix} = H(F,K) \begin{bmatrix} r \\ d \end{bmatrix},$$

where

$$H(F,K) = \begin{bmatrix} \frac{1}{1-FK} & \frac{F}{1-FK} \\ \frac{K}{1-FK} & \frac{1}{1-FK} \end{bmatrix}.$$

Given this, we have the following terminology. The transfer function $\frac{1}{1-FK}$ shall be referred to as the *sensitivity transfer function.* The transfer function $\frac{FK}{1-FK}$ shall be referred to as the *complementary sensitivity transfer function.* Finally, the transfer function $\frac{K}{1-FK}$ shall be referred to as the *reference to control transfer function.*

Let $\mathcal{R}$ denote some class of scalar functions which we shall refer to as the *stable* transfer functions (e.g. $\mathcal{H}^\infty$). Let $M(\mathcal{R})$ denote the set of all transfer function matrices with elements in $\mathcal{R}$.

We shall say that the feedback system in Figure 3.1 is *$\mathcal{R}$-internally stable* or that K *internally stabilizes F with respect to $\mathcal{R}$* if $H(F,K) \in M(\mathcal{R})$.

$S_{\mathcal{R}}(F)$ will be used to denote the set of all compensators K which internally stabilize the plant F with respect to $\mathcal{R}$. The class $\mathcal{R}$ should be viewed as a ring. We now define this and other algebraic concepts [56].

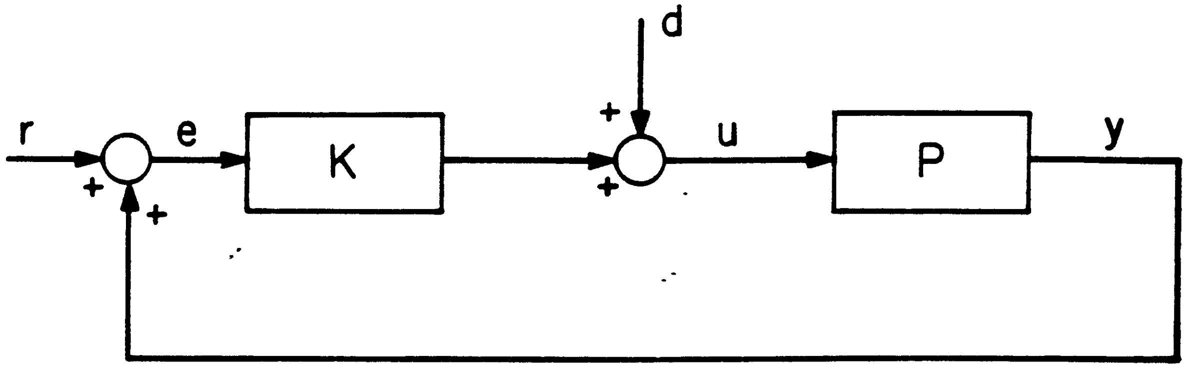

Figure 3.1: Feedback Control System Structure

Definition 3.2.1 (Algebraic Concepts)
A *ring* is a collection of elements $\mathcal{R}$ on which we define two binary operations $+ : \mathcal{R} \times \mathcal{R} \rightarrow \mathcal{R}$ and $\cdot : \mathcal{R} \times \mathcal{R} \rightarrow \mathcal{R}$. Given any $x, y, z \in \mathcal{R}$, the operations $+$ and $\cdot$ satisfy the following ring axioms.
(1) Closure: $x + y \in \mathcal{R}$.
(2) Commutivity: $x + y = y + x$.
(3) Associativity: $(x + y) + z = x + (y + z)$.
(4) There exists an element $0 \in \mathcal{R}$ called the *additive identity*, such that $x + 0 = x$.
(5) There exists an element $-x \in \mathcal{R}$ called the *additive inverse* of x, such that $x + (-x) = 0$.
(6) Closure: $x \cdot y \in \mathcal{R}$.
(7) Associativity: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
(8) There exists an element $1 \in \mathcal{R}$ called the *multiplicative identity*, such that $x \cdot 1 = 1 \cdot x = x$. The existence of such a multiplicative identity is what algebraists mean when they say that a ring $\mathcal{R}$ contains an identity.
(9) Distributivity: $x \cdot (y + z) = x \cdot y + x \cdot z$.
(10) Distributivity: $(x + y) \cdot z = x \cdot z + y \cdot z$.

We say that $\mathcal{R}$ is a *commutative* ring if the binary operation $(\cdot)$ is commutative on $\mathcal{R}$; i.e. $x \cdot y = y \cdot x$.

Let $\mathcal{R}$ be a commutative ring. We say that $\mathcal{R}$ is a *domain* if $x, y \neq 0$ implies that $x \cdot y \neq 0$ and $y \cdot x \neq 0$. x is a *unit* of y if $x \cdot y = 1$ and $y \cdot x = 1$. In such a case, we say that x and y are *invertible* in $\mathcal{R}$.

∎

Given this, we have the following parameterization for $S_{\mathcal{R}}(F)$ [56, pp.364], [58], [59]. We refer to it as the Youla et al. parameterization.

Proposition 3.2.1 (Youla et al. Parameterization)
Let $\mathcal{R}$ be a commutative domain with an identity. Let $\mathcal{F}(\mathcal{R})$ denote the fraction field associated with $\mathcal{R}$. Suppose $F \in \mathcal{F}(\mathcal{R})$. Suppose also that F has a coprime factorization (N_f, D_f) over the ring $\mathcal{R}$ with Bezout factors (N_k, D_k); i.e. suppose there exists elements $N_f, D_f, N_k, D_k \in \mathcal{R}$ such that

$$F = \frac{N_f}{D_f}$$

and

$$(D_f D_k - N_f N_k)^{-1} \in \mathcal{R}$$

for all s in the domain of definition of $\mathcal{R}$. Given this, the set of all compensators K which internally stabilize F with respect to $\mathcal{R}$ can be parameterized as follows:

$$S_{\mathcal{R}}(F) = \{\frac{N_k - D_f Q}{D_k - N_f Q} \mid Q \in \mathcal{R}\}^1.$$

■

Throughout the thesis, the symbol P shall denote an infinite dimensional plant [2] to be controlled. It shall always be assumed that P satisfies the assumptions of proposition 3.2.1. This shall be stated formally in assumption 4.2.1.

Given this, we shall often write $K(P, Q)$ to denote a particular compensator which stabilizes P. With this notatation, $K(P, \cdot)$ represents a *bijection* from $\mathcal{R}$ to $S_{\mathcal{R}}(P)$.

The utility of the above parameterization is that it allows us to express the individual transfer functions of $H(P, K(P, Q))$ as *affine functions* in the parameter Q:

$$H(P, K(P,Q)) = \begin{bmatrix} D_p(D_k - N_p Q) & N_p(D_k - N_p Q) \\ D_p(N_k - D_p Q) & D_p(D_k - N_p Q) \end{bmatrix}.$$

This affine dependence on Q is invaluable in optimization problems.

Finally, we note that if $P \in \mathcal{R}$, then we can choose $N_p = P$, $D_p = 1$, $N_k = 0$, and $D_k = 1$. The set of compensators K which internally stabilize P can then be parameterized as follows

$$K(P, Q) \stackrel{\text{def}}{=} \frac{-Q}{1 - PQ},$$

where Q can be any element in $\mathcal{R}$.

3.3 A Stability Result

In what follows, we shall require the following proposition on internal stability.

Proposition 3.3.1 (Internal Stability)
Let $\mathcal{R}$ denote a commutative domain with an identity. Let F denote a plant and C a compensator. Let $(N_f, D_f) \in \mathcal{R}$ be coprime factors for F over $\mathcal{R}$. Let $(N_c, D_c) \in \mathcal{R}$ be coprime factors for C over $\mathcal{R}$. We then have that C internally stabilizes F with respect to the ring $\mathcal{R}$ if and only if $\delta(F, C) \stackrel{\text{def}}{=} D_f D_c - N_f N_c$ is a unit of $\mathcal{R}$.

[1] It should be noted that the compensators in $S_{\mathcal{R}}(F)$ need not be causal [17, pp. 3], [?]. Conditions will be given so that this is avoided. All compensators which we construct in the thesis, of course, will be causal.

[2] By infinite dimensional we mean not necessarily finite dimensional.

■

Proof The proof in [56, pp. 45-46, lemma 4] applies to our general ring $\mathcal{R}$. The main ideas are now given.

If δ is a unit in $\mathcal{R}$, then the associated closed loop transfer function matrix

$$H(F,C) = \begin{bmatrix} D_f D_c & N_f D_c \\ D_f N_c & D_f D_c \end{bmatrix} \frac{1}{\delta}$$

has all of its transfer functions in $\mathcal{R}$; i.e. $H(F,C) \in M(\mathcal{R})$. This proves one direction.

If $H(F,C) \in M(\mathcal{R})$, then we have

$$\begin{bmatrix} D_f \\ N_f \end{bmatrix} \frac{1}{\delta} \begin{bmatrix} D_c \, N_c \end{bmatrix} \in M(\mathcal{R}).$$

The coprimeness of (N_f, D_f) and (N_c, D_c) then implies that $\frac{1}{\delta} \in \mathcal{R}$. This proves the other direction and hence completes the proof. ■

3.4 The Corona Theorem

We begin this section with an instructive example.

Example 3.4.1 ($\mathcal{H}^\infty$ as a Ring)
The function space $\mathcal{H}^\infty$ is a commutative domain with an identity [56]. It shall be used throughout the thesis to define stability; i.e. it will be the the only ring over which we will work.

Moreover, we shall always assume that our plant P belongs to the fraction field associated with $\mathcal{H}^\infty$. We shall also assume that it has a coprime factorization over the ring [3]. With this assumption, we will be able to use the parameterization in proposition 3.2.1. This, in turn, will allow us to express system transfer functions as affine functions with respect to the parameter Q.

When we deal with the ring $\mathcal{H}^\infty$ we are implicitly concerned with $\mathcal{L}^2$ - finite gain stability as defined in [11].

■

Given this, the following proposition will allow us to determine whether or not two functions in $\mathcal{H}^\infty$ are coprime.

Proposition 3.4.1 (Corona Theorem)
Given $N, D \in \mathcal{R}$ where $\mathcal{R}$ is $\mathcal{H}^\infty$, then N and D are coprime in $\mathcal{R}$ if and only if

$$\inf_{s \in C_+} |N(s)| + |D(s)| > 0.$$

■

This proposition can be found in [56, pp. 341-342]. It is sometimes referred to as the *Corona Theorem.* Loosely speaking, it says that N and D are coprime in $\mathcal{R}$ if and only if they have no common open right half plane zeros.

[3] It should be noted that not all functions in the fraction field of $\mathcal{A}$ have coprime factorizations [56]. In contrast, all stabilizable functions in the fraction field of $\mathcal{H}^\infty$ have coprime factorizations [52]. This shall receive further consideration in the sequel.

3.5 Summary

In this chapter we presented various results from the algebraic systems literature. We are now ready to precisely state what fundamental problems shall be addressed in this thesis.

Chapter 4

Statement of Fundamental Problems

4.1 Introduction

In this chapter we precisely define what we shall refer to as the *$\mathcal{N}$-Norm Approximate/Design J-Problem.* This problem shall be the focal point of the thesis. This problem addresses the questions: What is a "good" plant approximant? How do we obtain a near-optimal finite dimensional compensator?

We also define two additional problems, the *$\mathcal{N}$-Norm Purely Finite Dimensional J-Problem* and the *$\mathcal{N}$-Norm Loop Convergence J-Problem.* The first problem addresses the computation of optimal performance measures using finite dimensional techniques. The latter, examines the convergence properties of designs which result from the *Approximate/Design* approach taken in this thesis.

4.2 Basic Assumptions & Notation

In this section we shall make assumptions which will allow us to use the algebraic system-theoretic results of Chapter 3. In doing so we shall establish notation to be used throughout the thesis.

Let $P(s)$ denote the transfer function of a continuous time scalar infinite dimensional plant [1]. Also, let $\{P_n(s)\}_{n=1}^{\infty}$ denote a sequence of finite dimensional (real-rational) approximants of P. The sense in which P_n approximates P shall be stated in the sequel.

Throughout the thesis we shall assume that the plant P and the approximants P_n satisfy the assumptions in proposition 3.2.1. More precisely, we make the following assumption.

Assumption 4.2.1 (Permissible Plants and Approximants)
Let $\mathcal{F}(\mathcal{R})$ denote the fraction field of a *commutative domain* [2] $\mathcal{R}$ with an *identity* (e.g. $\mathcal{H}^{\infty}$). Let $\mathcal{F}_c(\mathcal{R})$ denote those elements of $\mathcal{F}(\mathcal{R})$ which are expressible as the ratio of coprime factors in $\mathcal{R}$. It shall be assumed that the plant P and the approximants $\{P_n(s)\}_{n=1}^{\infty}$ are elements of $\mathcal{F}_c(\mathcal{R})$.

∎

Given this, P has a coprime factorization over the ring $\mathcal{R}$; i.e. there exists elements $N_p, D_p, N_k, D_k \in \mathcal{R}$ such that

$$P = \frac{N_p}{D_p}$$

[1] By infinite dimensional we mean not necessarily finite dimensional. Although not explicitly stated, causality and linear time invariance are implied.

[2] A domain is a ring such that the product of non-zero elements is non-zero.

and

$$D_p D_k - N_p N_k = 1$$

for all s in the domain of definition of $\mathcal{R}$. It then follows from proposition 3.2.1 that the set of all compensators K which internally stabilize P with respect to the ring $\mathcal{R}$ can be parameterized as follows:

$$K(P,Q) = \frac{N_k - D_p Q}{D_k - N_p Q}$$

where $Q \in \mathcal{R}$. Given this, $K(P,\cdot)$ is a bijection from $\mathcal{R}$ to the set of all compensators $S_{\mathcal{R}}(P)$ which internally stabilize P with respect to $\mathcal{R}$.

Also, assumption 4.2.1 allows us to associate with each P_n the elements $N_{p_n}, D_{p_n}, N_{k_n}, D_{k_n} \in \mathcal{R}$ where

$$P_n = \frac{N_{p_n}}{D_{p_n}}$$

and

$$D_{p_n} D_{k_n} - N_{p_n} N_{k_n} = 1$$

for all s in the domain of definition of $\mathcal{R}$. Again, from proposition 3.2.1, it follows that the set of all compensators K_n which internally stabilize P_n with respect to $\mathcal{R}$ can be parameterized as follows:

$$K(P_n,Q) = \frac{N_{k_n} - D_{p_n} Q}{D_{k_n} - N_{p_n} Q}$$

where $Q \in \mathcal{R}$.

4.3 $\mathcal{N}$-Norm Approximate/Design J-Problem

In this section we precisely define the *$\mathcal{N}$-Norm Approximate/Design J-Problem.* First, we shall need some assumptions and definitions.

Given the discussion in the previous section, we let $J_{\mathcal{N}}(\cdot, K(\cdot,\cdot)) : \mathcal{F}_c(\mathcal{R}) \times \mathcal{F}_c(\mathcal{R}) \times \mathcal{R} \to R_+$ denote a *performance measure.* We shall refer to it as the *$\mathcal{N}$-norm J-measure.* This terminology is used since in the sequel, $\mathcal{N}$ will denote the norm on one of the function spaces ($\mathcal{H}^\infty$, $\mathcal{H}^2$) and J will be represent either a *sensitivity* criterion or a *mixed-sensitivity* criterion.

Given this, the *optimal performance* for the infinite dimensional plant P with respect to the *$\mathcal{N}$-norm J-measure* shall be defined as follows.

Definition 4.3.1 (Optimal Performance)

$$\mu_{opt} \stackrel{\text{def}}{=} \inf_{Q \in \mathcal{R}} J_{\mathcal{N}}(P, K(P,Q)).$$

■

The optimal [3] solution to this problem shall be denoted Q_{opt}. The corresponding compensator will be denoted K_{opt} and is given by

$$K_{opt} \stackrel{\text{def}}{=} K(P, Q_{opt}) = \frac{N_k - D_p Q_{opt}}{D_k - N_p Q_{opt}}.$$

[3] It may be that an optimal solution does not exist. This technical point is not important at the moment.

As stated earlier, our primary motive is that of finding a near-optimal finite dimensional compensator for the infinite dimensional plant P. To do so, one starts by examining the optimization problem defined in definition 4.3.1.

In general, definition 4.3.1 defines an *infinite dimensional optimization problem* which is very difficult to solve. The basic philosophy of this research endeavour has been to avoid the *Design/Approximate* approach which appears in the literature. In this approach, one first needs to solve the above infinite dimensional problem for an infinite dimensional compensator. Then, one approximates the resulting infinite dimensional compensator to get a desirable near-optimal finite dimensional compensator [14]-[16], [40]-[41]. Rather than this approach, we have taken an *Approximate/Design* approach.

In our *Approximate/Design* approach, we first "approximate" the plant P with a sequence of finite dimensional approximants $\{P_n\}_{n=1}^{\infty}$. Then, to obtain a suitable finite dimensional compensator, we consider the following *finite dimensional optimization problem.*

Definition 4.3.2 (Expected Performance)

$$\mu_n \stackrel{\text{def}}{=} \inf_{Q \in \mathcal{R}} J_{\mathcal{N}}(P_n, K(P_n, Q)).$$

∎

In the context of this work, we shall refer to μ_n as the *expected performance.* This terminology for μ_n is motivated by the fact that the numbers μ_n are typically used to guide us during the design process.

Let Q_n denote the optimal [4] solution for the problem in definition 5.5.1. In accordance with assumption 4.2.1, Q_n generates an internally stabilizing [5] compensator K_n for P_n. This compensator is given by:

$$K_n \stackrel{\text{def}}{=} K(P_n, Q_n) = \frac{N_{k_n} - D_{p_n} Q_n}{D_{k_n} - N_{p_n} Q_n}.$$

It is shown in section 6.3 that, in general K_n need not stabilize P, even when P_n is "close" to P in the uniform topology on $\mathcal{R}$ [6]. Consequently K_n, in general, cannot be placed in a feedback loop with the infinite dimensional plant P. It is thus appropriate to ask whether or not K_n can be modified to obtain a near-optimal finite dimensional compensator for P. We shall see that for a large class of problems, the answer to this question is affirmative.

Toward this end, we define a *"roll-off operator"*

$$r : Q_n \to \tilde{Q}_n$$

which maps Q_n to $\tilde{Q}_n \in \mathcal{R}$. From the above discussion, it is clear that the roll-off operator r must be chosen intelligently (see section 6.3). The exact form of r will be determined shortly.

Given this, we then consider the feedback system in Figure 4.1, obtained by substituting the finite dimensional compensator generated by $\tilde{Q}_n$, namely

$$\tilde{K}_n \stackrel{\text{def}}{=} K(P_n, \tilde{Q}_n) = \frac{N_{k_n} - D_{p_n} \tilde{Q}_n}{D_{k_n} - N_{p_n} \tilde{Q}_n}$$

[4] It may be that an optimal solution does not exist. Again, this technical point is not important at the moment.

[5] We note that this may not be the case since Q_n may not lie in the ring $\mathcal{R}$. This point is also associated with existence issues and is not critical at the moment.

[6] Since we shall be working over the ring $\mathcal{H}^{\infty}$, it is clear what is met by the uniform topology.

into a closed loop system with the infinite dimensional plant P. We note that although $\tilde{K}_n$ internally stabilizes P_n, it need not internally stabilize P. This critical issue will be addressed in the sequel.

Let $H(P, \tilde{K}_n)$ denote the resulting closed loop transfer function matrix from r, d to e, u. We then have

$$\begin{bmatrix} e \\ u \end{bmatrix} = H(P, \tilde{K}_n) \begin{bmatrix} r \\ d \end{bmatrix},$$

where

$$H(P, \tilde{K}_n) = \begin{bmatrix} \frac{1}{1-P\tilde{K}_n} & \frac{P}{1-P\tilde{K}_n} \\ \frac{\tilde{K}_n}{1-P\tilde{K}_n} & \frac{1}{1-P\tilde{K}_n} \end{bmatrix}.$$

Substituting for $\tilde{K}_n$ then gives

$$H(P, \tilde{K}_n(\tilde{Q}_n)) = \begin{bmatrix} D_p(D_{k_n} - N_{p_n}\tilde{Q}_n) & N_p(D_{k_n} - N_{p_n}\tilde{Q}_n) \\ D_p(N_{k_n} - D_{p_n}\tilde{Q}_n) & D_p(D_{k_n} - N_{p_n}\tilde{Q}_n) \end{bmatrix} \frac{1}{\delta(P, \tilde{K}_n(\tilde{Q}_n))}$$

where

$$\delta(P, \tilde{K}_n(\tilde{Q}_n)) \stackrel{\text{def}}{=} D_p(D_{k_n} - N_{p_n}\tilde{Q}_n) - N_p(N_{k_n} - D_{p_n}\tilde{Q}_n).$$

Assuming internal stability can be shown, we then have the following "natural" definition for the *actual performance.*

Definition 4.3.3 (Actual Performance)

$$\tilde{\mu}_n \stackrel{\text{def}}{=} J_{\mathcal{N}}(P, \tilde{K}_n).$$

■

Comment 4.3.1 (Internal Stability)
Since $\tilde{K}_n \stackrel{\text{def}}{=} N_{k_n} - D_{p_n}\tilde{Q}_n D_{k_n} - N_{p_n}\tilde{Q}_n$ internally stabilizes P_n, the factors $N_{k_n} - D_{p_n}\tilde{Q}_n$ and $D_{k_n} - N_{p_n}\tilde{Q}_n$ must be coprime. This follows from proposition 3.3.1. Given this, it follows from the same proposition, that $\tilde{K}_n(\tilde{Q}_n)$ internally stabilizes P if and only if $\frac{1}{\delta(P, \tilde{K}_n(\tilde{Q}_n))} \in \mathcal{R}$. This argument will be used in the sequel.

■

Given the above discussion, we define the *$\mathcal{N}$-Norm Approximate/Design J-Problem* as follows.

Problem 4.3.1 (Approximate/Design Problem)
Find conditions on the approximants $\{P_n\}_{n=1}^{\infty}$ and on the roll-off operator r such that

$$\lim_{n \to \infty} \tilde{\mu}_n = \mu_{opt}.$$

■

If a set of approximants $\{P_n\}_{n=1}^{\infty}$ satisfy the above condition, then we shall say that the approximants are "good"; i.e. a set of approximants $\{P_n\}_{n=1}^{\infty}$ are good if they allow us to satisfy the control objective. In this sense then, the *$\mathcal{N}$-Norm Approximate/Design J-Problem* addresses the question: What is a "good" finite dimensional approximant?

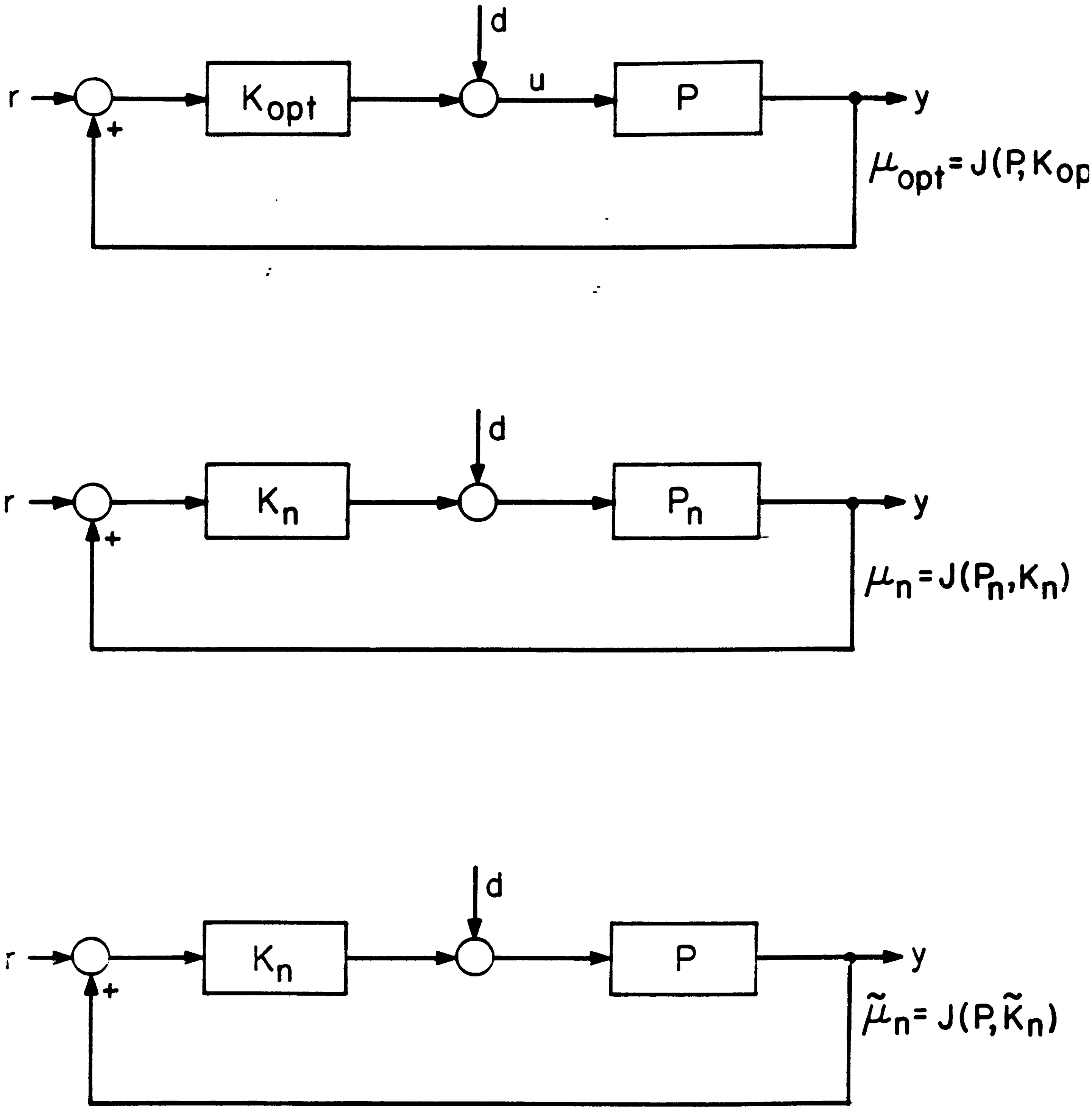

Figure 4.1: Visualization of Approximate/Design Methodology

Assuming $\tilde{K}_n$ internally stabilizes P, we have

$$\mu_{opt} \leq \tilde{\mu}_n.$$

This is because μ_{opt} is the *optimal performance.* Given this, we would like to show that

$$\lim_{n\to\infty} \tilde{\mu}_n \leq \mu_{opt}.$$

Doing so, however, is equivalent to finding an internally stabilizing finite dimensional compensator $\tilde{K}_n$ such that

$$\tilde{\mu}_n \leq \mu_{opt} + \epsilon$$

where $\epsilon > 0$ is a given (apriori) optimality tolerance. In this sense, the *$\mathcal{N}$-Norm Approximate/Design J-Problem* is equivalent to that of finding a near-optimal finite dimensional compensator for the infinite dimensional plant P.

This *$\mathcal{N}$-Norm Approximate/Design J-Problem* shall be the focal point of the thesis. More specifically, the problem shall be solved in the sequel for two distinct performance measures $J_{\mathcal{N}}$. To define these measures we shall need weighting functions $W, W_1, W_2 \in \mathcal{R}$.

Let $\mathcal{N}$ denote one of the function spaces $\mathcal{H}^\infty$ and $\mathcal{H}^2$. Suppose that $F, G \in \mathcal{F}_c(\mathcal{R})$ and $Q \in \mathcal{R}$. If the performance measure $J_{\mathcal{N}}$ has the form

$$J_{\mathcal{N}}(F, K(G,Q)) \stackrel{\text{def}}{=} \left\| \frac{W}{1 - FK(G,Q)} \right\|_{\mathcal{N}},$$

then problem 4.3.1 will be referred to as an *$\mathcal{N}$-Norm Approximate/Design Sensitivity Problem.* A solution to this problem shall be presented in Chapter 6 when the norm $\|\cdot\|_{\mathcal{N}}$ is the $\mathcal{H}^\infty$-norm. The $\mathcal{H}^2$-norm case is addressed in Chapter 8.

If the performance measure $J_{\mathcal{N}}$ has the form

$$J_{\mathcal{N}}(F, K(G,Q)) \stackrel{\text{def}}{=} \left\| \frac{\begin{bmatrix} W_1 \\ W_2 F K(G,Q) \end{bmatrix}}{1 - FK(G,Q)} \right\|_{\mathcal{N}},$$

or

$$J_{\mathcal{N}}(F, K(G,Q)) \stackrel{\text{def}}{=} \left\| \frac{\begin{bmatrix} W_1 \\ W_2 K(G,Q) \end{bmatrix}}{1 - FK(G,Q)} \right\|_{\mathcal{N}},$$

then problem 4.3.1 will be referred to as an *$\mathcal{N}$-Norm Approximate/Design Mixed-Sensitivity Problem.* A solution to this problem shall be presented in Chapter 7 when the norm $\|\cdot\|_{\mathcal{N}}$ is the $\mathcal{H}^\infty$-norm. The $\mathcal{H}^2$-norm case is addressed in Chapter 8.

The above *Approximate/Design* problems are difficult for several reasons. These reasons are discussed in section 6.3.

4.4 $\mathcal{N}$-Norm Purely Finite Dimensional J-Problem

In practice we would like to be able to compute μ_{opt} using finite dimensional algorithms. For the above sensitivity/mixed-sensitivity criteria, the literature has typically considered infinite dimensional eigenvalue/eigenfunction problems in order to compute μ_{opt} [14, pp. 28–31], [63, pp. 308]. With an ultimate intention of providing purely finite dimensional techniques for computing μ_{opt}, we shall also consider the following "purely" finite dimensional problem.

Problem 4.4.1 (Purely Finite Dimensional Problem)
Find conditions on the approximants $\{P_n\}_{n=1}^{\infty}$ *such that*

$$\lim_{n\to\infty} \mu_n = \mu_{opt}.$$

In the context of this work, we shall refer to this problem as the $\mathcal{N}$-Norm Purely Finite Dimensional J-Problem.

∎

This problem inherently addresses the question of computing the *optimal performance* μ_{opt} using finite dimensional techniques.

Again, if $J_{\mathcal{N}}$ is a sensitivity criterion then we will refer to this problem as the *$\mathcal{N}$-Norm Purely Finite Dimensional Sensitivity Problem.* Analogously, if $J_{\mathcal{N}}$ is a mixed-sensitivity criterion then we will refer to this problem as the *$\mathcal{N}$-Norm Purely Finite Dimensional Mixed-Sensitivity Problem.* These problems shall also be addressed in Chapters 6 and 7 for $\mathcal{H}^{\infty}$ criteria. Chapter 8 shall address the problem for the case of an $\mathcal{H}^2$ performance criteria.

4.5 $\mathcal{N}$-Norm Loop Convergence J-Problem

Having the *actual performance* $\tilde{\mu}_n$ and *expected performance* μ_n converge to the *optimal performance* μ_{opt} is quite desirable. However, given this it is still very important to understand in what sense the transfer functions associated with the *actual system* $(P, \tilde{K}_n)$ converge to those of the *optimal system* (P, K_{opt}). Given this, we define the *$\mathcal{N}$-Norm Loop Convergence J-Problem* as follows.

Problem 4.5.1 (Loop Convergence Problem)
Given that $\lim_{n\to\infty} \tilde{\mu}_n = \mu_{opt}$ *and* $\lim_{n\to\infty} \mu_n = \mu_{opt}$, *in what sense, if any, do the transfer functions associated with the* $(P, \tilde{K}_n)$*-loop converge to those of the optimal* (P, K_{opt})*-loop ?*

∎

In this problem we concern ourselves primarily with the convergence properties of designs which result from the *Approximate/Design* approach taken in this thesis.

This problem shall be addressed in the sequel for each of the afforementioned design criteria.

4.6 Summary

In this chapter three problems were defined: (1) The *$\mathcal{N}$-Norm Approximate/Design J-Problem,* (2) The *$\mathcal{N}$-Norm Purely Finite Dimensional J-Problem,* and (3) The *$\mathcal{N}$-Norm Loop Convergence J-Problem.*

In the sequel we shall address each of these problems for $\mathcal{H}^{\infty}$ and $\mathcal{H}^2$ sensitivity and mixed-sensitivity design criteria.

Chapter 5

$\mathcal{H}^\infty$ Model Matching Problem

5.1 Introduction

In this chapter we define the $\mathcal{H}^\infty$ *Model Matching Problem.* It is shown how near-optimal real-rational solutions can be constructed for this problem by considering sequences of appropriately formulated finite dimensional model matching problems. The constructions presented shall be heavily exploited in subsequent chapters on design via $\mathcal{H}^\infty$ optimization.

5.2 Infinite Dimensional $\mathcal{H}^\infty$ Model Matching Problem

In subsequent chapters, we will be examining optimization problems which have the following structure.

Definition 5.2.1

$$\mu_o \stackrel{\text{def}}{=} \inf_{Q \in \mathcal{H}_0^\infty} \|T_1 - T_2 Q\|_{\mathcal{H}^\infty}$$

where $T_1, T_2 \in \mathcal{H}^\infty$.

Given this, the $\mathcal{H}^\infty$ *Model Matching Problem is that of finding a near-optimal* Q*-parameter in* $\mathcal{H}_0^\infty$.

■

Since $T_2 \in \mathcal{H}^\infty$, we know from proposition 2.7.6 that it possesses an *inner-outer factorization* over $\mathcal{H}^\infty$. We will denote this factorization by $T_2 = T_{2_i} T_{2_o}$, where T_{2_i} is *inner* in $\mathcal{H}^\infty$ and T_{2_o} is *outer* in $\mathcal{H}^\infty$. This factorization shall be exploited throughout the chapter.

Throughout this chapter, the following technical assumption shall be made on T_1 and T_2.

Assumption 5.2.1 (T_1 and T_2)
(1) $T_1 \in \mathcal{C}_e$ [1].
(2) T_{2_o} has a finite number of zeros on the extended imaginary axis; each with finite algebraic multiplicity. See comment 5.2.1 below.
(3) If T_{2_o} is not invertible in $\mathcal{H}^\infty$, then we assume that $T_{2_o} \in \mathcal{C}_e$.

---

[1] Functions in $\mathcal{C}_e$ are continuous everywhere on the extended imaginary axis.

■

The following comment establishes notation regarding the imaginary zeros of T_{2_o}. It also clarifies what we mean by each zero having finite algebraic multiplicity.

Comment 5.2.1 (Imaginary Zeros of T_{2_o}: Finite Algebraic Multiplicity)
We shall denote the zeros of T_{2_o} on the extended imaginary axis by the finite sequence $\{\omega_k\}_{k=0}^{l}$. It shall be assumed that the sequence is increasing and that its elements are distinct, non-negative, extended real numbers. We shall also adopt the convention that $\omega_0 = 0$ and $\omega_l = \infty$. Given this, $\{\omega_k\}_{k=1}^{l-1}$ will denote finite zeros of T_{2_o} on the positive imaginary axis. $\{-\omega_k\}_{k=1}^{l-1}$ will denote finite zeros of T_{2_o} on the negative imaginary axis.

By assumption 5.2.1, each ω_k has finite algebraic multiplicity. This multiplicity shall be denoted m_k, is non-negative, and is generally a fraction.

Since $T_{2_o} \in \mathcal{C}_e$, we know that it is bounded away from zero on all closed frequency intervals not containing an imaginary zero. Consequently, if we define

$$F(s) \stackrel{\text{def}}{=} s^{m_0} \prod_{k=1}^{l-1} (s^2 + \omega_k^2)^{m_k}$$

and

$$G_a(s) \stackrel{\text{def}}{=} \left(\frac{a}{s+a}\right)^{m_l}$$

where $a \in R_+$, then

$$T_{2_o}^{-1}(\cdot)\, F(\cdot)\, F^{-1}(\cdot + \frac{1}{a})\, G_a(\cdot), \quad T_{2_o}(\cdot)\, F^{-1}(\cdot)\, G_a^{-1}(\cdot) \in \mathcal{H}^\infty \cap \mathcal{C}_e$$

for each $a \in R_+$.

■

Several points need to be made in order to emphasize the broad applicability of assumption 5.2.1 above.

Comment 5.2.2 (Broad Applicability)

(1) T_1 and T_2 may be real-rational.

(2) T_{2_i} need not be continuous on the extended imaginary axis as assumed in [60]. It may, for example, have a *singular inner part*; e.g. $T_{2_i} = e^{-s}$.

(3) T_{2_i} may have an *infinite Blaschke product* [30, pp. 132]; e.g.

$$T_{2_i} = \prod_{k=1}^{\infty} \frac{|1 - z_k^2|}{1 - z_k^2} \; \frac{s - z_k}{s + \overline{z_k}}$$

where $\{z_k\}_{k=1}^{\infty}$ are open right half plane zeros satisfying the *Blaschke condition*

$$\sum_{k=1}^{\infty} \frac{Re(z_k)}{1 + |z_k|^2} < \infty.$$

(4) The zeros of T_{2_o} on the extended imaginary axis, strictly speaking, may be branch points of T_{2_o} with respect to the entire complex plane; e.g. $T_{2_o} = \sqrt{\frac{s}{s+1}}$. In such a case we assume a single-valued analytic branch in the open right half plane.

(5) T_2 may possess removable singularities, poles, essential singularities, and branch points in the open left half plane.

(6) Although $T_{2_o} \in \mathcal{C}_e$, this does not mean that $T_2 \in \mathcal{C}_e$; e.g. $T_2 = e^{-s}\left(\frac{s-1}{s+1}\right)\left(\frac{s+2}{s+3}\right)$.

(7) Although the outer function $T_{2_o} = \frac{1}{1-\frac{1}{2}e^{-s}}$ is not continuous on the extended imaginary axis, it is admissible since it is invertible in $\mathcal{H}^\infty$; i.e. $1 - e^{-s} \in \mathcal{H}^\infty$.

(8) Cases in which the associated Hankel operator $\Gamma_{T_1 T_{2_i}^*}$ is non-compact are permissible; e.g. $T_1 = \frac{s+1}{s+\beta}$, $T_{2_i} = e^{-s}$ (cf. proposition 2.9.2).

∎

The above *Model Matching Problem* should be viewed as an *inversion problem.* This is because in posing the problem we indirectly tell the mathematics to find an admissible Q which inverts as much of T_2 as possible. The problem should thus be approached with an "inversion mentality".

In this section we shall construct a sequence $\{Q_m\}_{m=1}^{\infty} \subset \mathcal{H}_0^\infty$ such that given an *optimality tolerance* $\epsilon > 0$, however small,

$$\mu_o \leq \|T_1 - T_2 Q_m\|_{\mathcal{H}^\infty} \leq \mu_o + \epsilon$$

for m sufficiently large. This will allow us to obtain a *near-optimal* Q-parameter and thus provide a solution to the $\mathcal{H}^\infty$ *Model Matching Problem.* To construct this sequence, we first shed light on exactly what part of T_2 should be inverted.

From the above inner-outer factorization for T_2, we immediately note that, in general, T_{2_i} is not invertible in $\mathcal{H}^\infty$. We also note that any zeros which T_{2_o} may have on the imaginary axis, are also non-invertible. This leads us to conjecture that we should invert T_{2_o} *away from its imaginary zeros in the closed right half plane.* This statement/conjecture shall be formalized and confirmed shortly.

In order to gain insight into the $\mathcal{H}^\infty$ *Model Matching Problem*, one usually starts with some version of the following proposition [23], [44, pp. 46]. The proposition emphasizes (and quantifies) the non-invertibility of T_{2_i}.

Proposition 5.2.1 (Inner Problem)
Each of the following are equal:

$$\inf_{Z \in \mathcal{H}^\infty} \|T_1 - T_{2_i} Z\|_{\mathcal{H}^\infty} \tag{1}$$

$$\left\|\Gamma_{T_1 T_{2_i}^*}\right\| \tag{2}$$

Moreover, there exists $Z_o \in \mathcal{H}^\infty$, not necessarily unique, which attains the infimum in the "inner problem" defined by (1); i.e.

$$\|T_1 - T_{2_i} Z_o\|_{\mathcal{H}^\infty} = \min_{Z \in \mathcal{H}^\infty} \|T_1 - T_{2_i} Z\|_{\mathcal{H}^\infty}.$$

In the sequel, we shall refer to Z_0 as the "inner solution".

Proof The existence of an infimizer Z_o can be proved using standard weak* convergence arguments [5, pp. 134-137], [35, pp. 128], [24, pp. 85 - 86]. The fact that the infimum is $\left\|\Gamma_{T_1 T_{2_i}^*}\right\|$ is essentially a special case of Nehari's Theorem [38]. Elements of this proposition can also be found in [1], [39], [46], [51]. ∎

Typically, one solves the "inner problem", defined in proposition 5.2.1, in order to get a handle on the the optimization problem defined in definition 5.2.1.

The following lemma gives us further insight about the $\mathcal{H}^\infty$ *Model Matching Problem.* More specifically, it gives us a lower bound on μ_o which supports our conjecture that we should invert T_{2_o}, away from its imaginary zeros, in the closed right half plane.

Lemma 5.2.1 (Bounds on μ_o)

$$\max\{|T_1(j\infty)|,\ \max_k |T_1(j\omega_k)|,\ \left\|\Gamma_{T_1 T_{2_i}^*}\right\|\} \le \ \mu_o \ \le \ \|T_1\|_{\mathcal{H}^\infty}$$

∎

Proof To prove this, we begin by noting that

$$\mu_o \stackrel{\text{def}}{=} \inf_{Q\in\mathcal{H}_0^\infty} \|T_1 - T_2 Q\|_{\mathcal{H}^\infty} \le \|T_1\|_{\mathcal{H}^\infty}$$

since $0 \in \mathcal{H}_0^\infty$. This proves the upper inequality in the lemma.

To prove the lower inequality, it will suffice to show that each of the terms $|T_1(j\infty)|$, $\max_k |T_1(j\omega_k)|$, and $\left\|\Gamma_{T_1 T_{2_i}^*}\right\|$ is a lower bound for μ_o.

From the maximum modulus theorem for $\mathcal{H}^\infty$ (proposition 2.7.2), it follows that $|T_1(j\infty)| \le \|T_1 - T_2 Q\|_{\mathcal{H}^\infty}$ for any $Q \in \mathcal{H}_0^\infty$. It thus follows that

$$|T_1(j\infty)| \le \mu_o.$$

Since $T_2(j\omega_k) = 0$ for each k, from proposition 2.7.2 we have $\max_k |T_1(j\omega_k)| \le \|T_1 - T_2 Q\|_{\mathcal{H}^\infty}$ for any $Q \in \mathcal{H}_0^\infty$. It thus follows that

$$\max_k |T_1(j\omega_k)| \le \mu_o.$$

Since $T_{2_o}\mathcal{H}_0^\infty \subset \mathcal{H}^\infty$, we have $\inf_{Z\in\mathcal{H}^\infty} \|T_1 - T_{2_i} Z\|_{\mathcal{H}^\infty} \le \|T_1 - T_{2_i} T_{2_o} Q\|_{\mathcal{H}^\infty}$ for each $Q \in \mathcal{H}_0^\infty$. This then implies that $\inf_{Z\in\mathcal{H}^\infty} \|T_1 - T_{2_i} Z\|_{\mathcal{H}^\infty} \le \mu_o$. By proposition 5.2.1 the left hand side of this inequality equals $\left\|\Gamma_{T_1 T_{2_i}^*}\right\|$. We thus have

$$\left\|\Gamma_{T_1 T_{2_i}^*}\right\| \le \mu_o.$$

Combining the above gives us the desired lower inequality:

$$\max\{|T_1(j\infty)|,\ \max_k |T_1(j\omega_k)|,\ \left\|\Gamma_{T_1 T_{2_i}^*}\right\|\} \le \mu_o.$$

This completes the proof. ∎

Comment 5.2.3 (Lower bound for μ_o at ∞)
Note that if we were infimizing over $\mathcal{H}^\infty$, rather than $\mathcal{H}_0^\infty$, then we would again have the inequality $|T_1(j\infty)| \le \mu_o$, if T_2 were to roll-off.

∎

We see that lemma 5.2.1 gives a lower bound on μ_o which depends on T_{2_i} and on the imaginary zeros of T_{2_o}. This supports our conjecture that we should invert T_{2_o}, away from its imaginary zeros, in the closed right half plane. We shall soon show that the above lower bound is actually equal to μ_o. The lemma also shows that since $T_1 \in \mathcal{H}^\infty$, μ_o will necessarily be finite.

Our interest in the "inner problem" will be to use its solution Z_o to construct near-optimal solutions for the $\mathcal{H}^\infty$ *Model Matching Problem.* The idea will be to appropriately modify ("roll-off") $T_{2_o}^{-1} Z_o$ so that the resulting function is admissible [2] and near-optimal. This modification shall be carried out in two steps. Each step is now described.

Main Ideas: Construction of $\{Q_{m,n}\}$.

The first step will be to construct a sequence $\{f_m\}_{m=1}^\infty \subset \mathcal{H}_0^\infty$ to *modify* the "inner solution". By this we mean that f_m will be such that

$$\|T_1 - T_{2_i} Z_o f_M\|_{\mathcal{H}^\infty} \leq \mu_o + 2\epsilon$$

for M sufficiently large. The function f_m, as we shall see, will contain all the essential information about the imaginary axis zeros of T_{2_o}. It will be small near each imaginary zero of T_{2_o} and it will approximate unity elsewhere. To construct the function, we shall exploit the fact that T_{2_0} has a finite number of zeros on the extended imaginary axis. The phase properties of f_m will be critical in achieving the above inequality.

After constructing the sequence $\{f_m\}_{m=1}^\infty$ to modify the "inner solution", we shall construct an *inverting sequence* $\{g_n\}_{n=1}^\infty \subset \mathcal{H}^\infty$ for T_{2_o}. This terminology will be justified shortly. The sequence will be such that

$$T_{2_o}{}^{-1} g_n \in \mathcal{H}^\infty$$

and

$$\lim_{n\to\infty} \|T_{2_i} Z_o f_M (1 - g_n)\|_{\mathcal{H}^\infty} = 0$$

for each $M, n \in Z_+$.

We then define a double sequence according to

$$Q_{m,n} \stackrel{\text{def}}{=} T_{2_o}{}^{-1}\, Z_o\, f_m\, g_n.$$

This sequence will lie in $\mathcal{H}_0^\infty$ for each $m, n \in Z_+$. Hence, it will be admissible.

Finally, the above are combined with the following inequality

$$\|T_1 - T_2 Q_{M,n}\|_{\mathcal{H}^\infty} \leq \|T_1 - T_{2_i} Z_o f_M\|_{\mathcal{H}^\infty} + \|T_{2_i} f_M (1 - g_n)\|_{\mathcal{H}^\infty},$$

to show that there exists $M, N \in Z_+$ such that

$$\mu_o \leq \|T_1 - T_2 Q_{M,n}\|_{\mathcal{H}^\infty} \leq \mu_o + 3\epsilon$$

for all $n \geq N$.

We thus see that, on the one hand, the sequence $\{f_m\}_{m=1}^\infty$ allows us to keep the essential information contained in T_{2_o}. On the other hand, the sequence $\{g_n\}_{n=1}^\infty$ allows us to invert that part of T_{2_o} which is not essential. The two sequence together shall implement our idea of *invertng T_{2_o}, away from its imaginary zeros, in the closed right half plane.* ∎

The following algebraic result will shed light on the construction of the sequence $\{f_m\}_{m=1}^\infty$ which will modify the inner solution.

[2] In general, $T_{2_o}{}^{-1} Z_o$ will have poles on the extended imaginary axis.

Lemma 5.2.2 (Algebraic Result)
Suppose $T, Y, f \in \mathcal{H}^\infty$ where $\|f\|_{\mathcal{H}^\infty} \le 1$ and $\|Y\|_{\mathcal{H}^\infty} \le \|T\|_{\mathcal{H}^\infty}$. It then follows that

$$|T + (Y-T)f|^2{}_{[s=j\omega]} \le \max\{|T(j\omega)|^2, \|Y\|^2_{\mathcal{H}^\infty}\} + 2\|T\|^2_{\mathcal{H}^\infty}\{2|1 - cos\theta_f| + |sin\theta_f - \theta_f| + |\theta_f|\}$$

(almost) everywhere on the extended imaginary axis.

■

Proof The proof of this lemma is purely algebraic and is given to accommodate the skeptical.

$$|T + (Y-T)f|^2{}_{[s=j\omega]} = |T(1-f) + Yf|^2{}_{[s=j\omega]}$$

$$= |T(1-f)|^2 + T^*(1-f^*)Yf + T(1-f)Y^*f^* + |Yf|^2$$

$$= |T|^2(1 - f - f^* + |f|^2) + T^*Yf - T^*Y|f|^2 + TY^*f^* - TY^*|f|^2 + |Yf|^2$$

$$= |T|^2(1 - 2|f| + |f|^2) + 2|T|^2|f| - |T|^2(f + f^*) + T^*Yf + TY^*f^* - |f|^2(T^*Y + TY^*) + |Yf|^2$$

$$= |T|^2(1-|f|)^2 + 2|T|^2|f|(1-\cos\theta_f) + T^*Y(f-f^*) + (T^*Y + TY^*)f^* - 2|T||Y||f|^2\cos(\theta_t - \theta_y) + |Yf|^2$$

$$= |T|^2(1-|f|)^2 + 2|T|^2|f|(1-\cos\theta_f) + |Yf|^2 - 2|T||Y||f|^2\cos(\theta_t - \theta_y) + |T||Y|e^{j(\theta_y - \theta_t)}j2|f|\sin\theta_f$$
$$+2|T||Y|\cos(\theta_t - \theta_y)|f|e^{-j\theta_f}$$

After expanding each complex exponential into real and imaginary components, one obtains

$$|T + (Y-T)f|^2{}_{[s=j\omega]} = |T|^2(1-|f|)^2 + |Yf|^2 + 2|T||Y||f|\cos(\theta_t - \theta_y)(1-|f|)$$

$$+2|T|^2|f|(1-\cos\theta_f) + 2|T||Y||f|\cos(\theta_t - \theta_y)(cos\theta_f - 1)$$

$$+2|T||Y||f|\sin(\theta_t - \theta_y)(\sin\theta_f - \theta_f)$$

$$+2|T||Y||f|\sin(\theta_t - \theta_y)\theta_f$$

(almost) everywhere on the extended imaginary axis.

Given this, let

$$\mu \stackrel{\text{def}}{=} \max\{|T(j\omega)|, \|Y\|_{\mathcal{H}^\infty}\}.$$

With this definition, and the fact that $\|f\|_{\mathcal{H}^\infty} \le 1$, we then have

$$|T + (Y-T)f|^2{}_{[s=j\omega]} \le \mu^2(1 - 2|f| + |f|^2 + |f|^2 + 2|f| - 2|f|^2)$$

$$+4\mu^2|1 - cos\theta_f| + 2\mu^2|sin\theta_f - \theta_f| + 2\mu^2|\theta_f|$$

or equivalently,

$$|T + (Y-T)f|^2{}_{[s=j\omega]} \le \mu^2 + 4\mu^2|1 - cos\theta_f| + 2\mu^2|sin\theta_f - \theta_f| + 2\mu^2|\theta_f|$$

(almost) everywhere on the extended imaginary axis.

Since by assumption $\|Y\|_{\mathcal{H}^\infty} \le \|T\|_{\mathcal{H}^\infty}$, we have $\mu \le \|T\|_{\mathcal{H}^\infty}$. This, and the above inequality, then gives us that

$$|T + (Y-T)f|^2{}_{[s=j\omega]} \le \mu^2 + 2\|T\|^2_{\mathcal{H}^\infty}\{2|1 - cos\theta_f| + |sin\theta_f - \theta_f| + |\theta_f|\}$$

(almost) everywhere on the extended imaginary axis. This, however, is the desired result. ■

Comment 5.2.4 (A Convexity Argument)
If we let $\theta_f = 0$ in the above lemma, we get $f \in [0, 1]$. In such a case, the desired inequality follows from a simple convexity argument:

$$|T + (Y - T)f| \le (1 - f)|T| + f|Y| \le \max\{|T|, |Y|\}.$$

This simple case captures the essential ingredient needed to construct the sequence $\{f_m\}_{m=1}^{\infty}$. This is demonstrated in the following proposition.

■

Proposition 5.2.2 (Phase Result)
Let $\{f_m\}_{m=1}^{\infty}$ denote a sequence of $\mathcal{H}^\infty$ functions such that

$$\|f_m\|_{\mathcal{H}^\infty} \le 1$$

for each $m \in Z_+$ and

$$\lim_{m\to\infty} \|\theta_{f_m}\|_{\mathcal{L}^\infty} = 0.$$

Given this, there exists $M \stackrel{\text{def}}{=} M(\epsilon, \|T_1\|_{\mathcal{H}^\infty}) \in Z_+$ (independent of ω) such that

$$|T_1 - T_{2_i} Z_o f_m|_{[s=j\omega]} \le \max\{|T_1(j\omega)|, \left\|\Gamma_{T_1 T_{2_i}^*}\right\|\} + \epsilon$$

for all $m \ge M$, (almost) everywhere on the extended imaginary axis.

■

Proof We begin by defining $Y \stackrel{\text{def}}{=} T_1 - T_{2_i} Z_o$. This then gives

$$T_1 - T_{2_i} Z_o f_m = T_1 + (Y - T_1) f_m.$$

From proposition 5.2.1, $\|Y\|_{\mathcal{H}^\infty} = \left\|\Gamma_{T_1 T_{2_i}^*}\right\|$. From lemma 5.2.1, $\left\|\Gamma_{T_1 T_{2_i}^*}\right\| \le \|T_1\|_{\mathcal{H}^\infty}$. Combining these gives $\|Y\|_{\mathcal{H}^\infty} \le \|T_1\|_{\mathcal{H}^\infty}$. Given this, from the algebraic result in lemma 5.2.2, it follows that

$$|T_1 - T_{2_i} Z_o f_m|^2{}_{[s=j\omega]} = |T_1 + (Y - T_1) f_m|^2{}_{[s=j\omega]}$$

$$\le \max\{|T_1(j\omega)|^2, \|Y\|_{\mathcal{H}^\infty}^2\} + 2\|T_1\|_{\mathcal{H}^\infty}^2 \{2|1 - cos\theta_{f_m}| + |sin\theta_{f_m} - \theta_{f_m}| + |\theta_{f_m}|\}$$

(almost) everywhere on the extended imaginary axis. Substituting $\|Y\|_{\mathcal{H}^\infty} = \left\|\Gamma_{T_1 T_{2_i}^*}\right\|$ into the first term then gives

$$|T_1 - T_{2_i} Z_o f_m|^2{}_{[s=j\omega]} \le \max\{|T_1(j\omega)|^2, \left\|\Gamma_{T_1 T_{2_i}^*}\right\|^2\} + 2\|T_1\|_{\mathcal{H}^\infty}^2 \{2|1 - cos\theta_{f_m}| + |sin\theta_{f_m} - \theta_{f_m}| + |\theta_{f_m}|\}$$

(almost) everywhere on the extended imaginary axis. From the assumed phase property of f_m, it follows that the last three terms can be made arbitrarily small, uniformly in frequency, by taking m sufficiently large. More precisely, there exists $M \stackrel{\text{def}}{=} M(\epsilon, \|T_1\|_{\mathcal{H}^\infty}) \in Z_+$ (independent of ω) such that

$$|T_1 - T_{2_i} Z_o f_m|_{[s=j\omega]} \le \max\{|T_1(j\omega)|, \left\|\Gamma_{T_1 T_{2_i}^*}\right\|\} + \epsilon$$

for all $m \ge M$, (almost) everywhere on the extended imaginary axis. This completes the proof. ■

We emphasize that only magnitude and phase assumptions on f_m were used to obtain the above result. That such assumptions should suffice follows "intuitively" from the simple convexity argument given in comment 5.2.4.

In the proof of proposition 5.2.2 above, the phase assumption seemed indespensable. The following shows that it can sometimes be relaxed.

Comment 5.2.5 (Relaxing the Phase Assumption)
It should be noted that if $T_1(j\omega) = 0$, then the phase assumption on f_m becomes unnecessary. Although not transparent from the proofs given, it can be seen as follows. If $|f_m| \leq 1$ and $T_1(j\omega) = 0$, then we have

$$|T_1 - T_{2_i} Z_o f_m|_{[s=j\omega]} = |Y f_m|_{[s=j\omega]} \leq |Y|_{[s=j\omega]} \leq \|Y\|_{\mathcal{H}^\infty} \leq \left\|\Gamma_{T_1 T_{2_i}^*}\right\|,$$

which proves the claim.

■

We now construct the sequence $\{f_m\}_{m=1}^{\infty}$. It will possess the properties needed to *modify* the inner solution Z_o, in the manner suggested earlier. To perform the construction, we exploit the fact that T_{2_o} has a finite number of zeros on the extended imaginary axis.

Proposition 5.2.3 (Sequence to Modify Inner Solution)
There exists a sequence $\{f_m\}_{m=1}^{\infty} \subset \mathcal{H}_0^\infty$ of outer functions which possess the following properties:

(1) $f_m \in \mathcal{C}_e$ for each $m \in Z_+$.
(2) $\|f_m\|_{\mathcal{H}^\infty} \leq 1$ for each $m \in Z_+$.
(3) The phase of f_m can be made arbitrarily small, uniformly in frequency, by taking m sufficiently large; i.e.

$$\lim_{m\to\infty} \|\theta_{f_m}\|_{\mathcal{L}^\infty} = 0.$$

(4) f_m converges uniformly to unity on all compact frequency intervals excluding the points $\{\omega_k\}_{k=0}^{l}$; i.e. (see definition 2.10.5) for each $\delta > 0$, however small,

$$\lim_{m\to\infty} \left\|(1 - f_m) X_{R/B(\infty,\delta)\cup\cup_{k=0}^{l} B(\omega_k,\delta)}\right\|_{\mathcal{H}^\infty} = 0.$$

(5) If $m_k > 0$, then $f_m(\pm j\omega_k) = 0$ for each $m \in Z_+$.

■

Proof
The sequence defined by the following irrational function satisfies conditions (1)-(5) of the proposition.

$$f_m(s) \stackrel{\text{def}}{=} \left(\frac{s}{s+\frac{1}{m}}\right)^{\frac{m_0}{m}} \prod_{k=1}^{l-1} \left(\frac{s^2+\omega_k^2}{s^2+2\frac{1}{m}s+\omega_k^2+\frac{1}{m}^2}\right)^{\frac{m_k}{m}} \left(\frac{m}{s+m}\right)^{\frac{m_l+1}{m}}.$$

We note, in particular, that

$$\|\theta_{f_m}\|_{\mathcal{L}^\infty} \leq \frac{B\pi}{2m}$$

for some $B \in R_+$. We see that f_m is irrational even when the m_k are integers. We also note that

$$f_m(s) = [\, F(s)\, F^{-1}(s+\frac{1}{m})\, G_m(s) \left(\frac{m}{s+m}\right)]^{\frac{1}{m}}.$$

This completes the construction. ■

Comment 5.2.6 (Phase is Critical)
It should be emphasized that it is the phase property of f_m that makes it a very special function. It is easy to construct rational sequences which satisfy the other properties in proposition 5.2.3; e.g. the sequence defined by

$$\tilde{f}_m(s) \stackrel{\text{def}}{=} \left(\frac{s}{s+\frac{1}{m}}\right) \prod_{k=1}^{l-1} \left(\frac{s^2+\omega_k^2}{s^2+2\frac{1}{m}s+\omega_k^2+\frac{1}{m}^2}\right) \left(\frac{m}{s+m}\right).$$

However, finding a rational sequence which also possesses the special phase properties, at first glance, seems very difficult. Since $f_m \in \mathcal{C}_e$, we know from proposition 2.10.1, that such a sequence can be constructed.

∎

We now use the sequence $\{f_m\}_{m=1}^{\infty}$ from proposition 5.2.3, in conjunction with proposition 5.2.2, to obtain the following key theorem. To our knowledge, the key ideas behind the following theorem first appeared in [14, pp. 68-72].

Theorem 5.2.1 (Modifying the Inner Solution)
Given that $T_1 \in \mathcal{C}_e$ and that T_{2_o} has a finite number of zeros on the extended imaginary axis, there exists $M \stackrel{\text{def}}{=} M(\epsilon, T_1) \in Z_+$ such that

$$\|T_1 - T_{2_i} Z_o f_m\|_{\mathcal{H}^\infty} \leq \mu_o + 2\epsilon$$

for all $m \geq M$.

∎

Proof To prove the theorem, we begin by defining

$$Y \stackrel{\text{def}}{=} T_1 - T_{2_i} Z_o.$$

This then gives

$$Y_m \stackrel{\text{def}}{=} T_1 - T_{2_i} Z_o f_m = T_1 + (Y - T_1) f_m.$$

We shall examine the magnitude of this quantity on two sets. We denote these sets S_δ and R/S_δ. To prove the theorem we proceed in three steps. (1) First, we show that there exists a set S_δ on which $|Y_m|$ satisfies the inequality for m sufficiently large. Here we exploit the continuity of T_1 and the magnitude and phase properties of f_m (modulo proposition 5.2.3). (2) We then exploit the magnitude and compact convergence properties of f_m to show that $|Y_m|$ will satisfy the inequality on R/S_δ for m sufficiently large. (3) Finally, we combine the results in steps (1) and (2).

Step 1: Analysis on S_δ.
In this step we construct a set S_δ such that

$$\|Y_m X_{S_\delta}\|_{\mathcal{H}^\infty} \leq \mu_o + 2\epsilon.$$

To do this we proceed as follows.

(a) Constructing S_δ.

First consider the behavior of T_1 near $j\infty$ [3]. By assumption 5.2.1, T_1 is continuous everywhere on the extended imaginary axis and hence at $j\infty$. It thus follows that there exists $\Omega \stackrel{\text{def}}{=} \Omega(\epsilon, T_1) \in R_+$ such that $|T_1(j\omega) - T_1(j\infty)| \leq \epsilon$ for all ω such that $|\omega| \geq \Omega$; i.e. for each ω in $B(\infty, \frac{1}{\Omega}) \stackrel{\text{def}}{=} (-\infty, -\Omega) \cup (\Omega, \infty)$. Hence

$$\left\| (T_1 - T_1(j\infty)) X_{B(\infty, \frac{1}{\Omega})} \right\|_{\mathcal{H}^\infty} \leq \epsilon.$$

This takes care of the point at ∞. Without loss of generality, we can assume that the points $\{\omega_k\}_{k=0}^{l}$ are finite.

Now consider the behavior of T_1 near each point $j\omega_k$ [4]. Since T_1 is continuous at $j\omega_k$, it follows that there exists $\delta_k \stackrel{\text{def}}{=} \delta_k(\epsilon, T_1) \in R_+$ such that $|T_1(j\omega) - T_1(j\omega_k)| \leq \epsilon$ for all ω in $B(\omega_k, \delta_k) \stackrel{\text{def}}{=} (-\omega_k - \delta_k, -\omega_k + \delta_k) \cup (\omega_k - \delta_k, \omega_k + \delta_k)$. Hence

$$\left\| (T_1 - T_1(j\omega_k)) X_{B(\omega_k, \delta_k)} \right\|_{\mathcal{H}^\infty} \leq \epsilon$$

for each k. This takes care of each point ω_k.

We now define

$$\delta \stackrel{\text{def}}{=} \delta(\epsilon, T_1) \stackrel{\text{def}}{=} \min\{\frac{1}{\Omega}, \min_k \delta_k\}.$$

This quantity is well defined, and positive, since we are minimizing over a finite number of positive objects. This is because T_{2_o} has a finite number of zeros. Given this, it then follows that

$$\left\| (T_1 - T_1(j\infty)) X_{B(\infty, \delta)} \right\|_{\mathcal{H}^\infty} \leq \epsilon$$

and

$$\left\| (T_1 - T_1(j\omega_k)) X_{B(\omega_k, \delta)} \right\|_{\mathcal{H}^\infty} \leq \epsilon$$

for each k.

We now define the set S_δ as follows

$$S_\delta \stackrel{\text{def}}{=} B(\infty, \delta) \cup \cup_{k=0}^{l} B(\omega_k, \delta).$$

This set represents points which are "near" the points $\{\infty, \{\omega_k\}_{k=0}^{l}\}$.

(b) Upperbound for T_1 on S_δ.

Here, we would like to obtain an upper bound for T_1 on S_δ. From part (a), it follows that

$$\left\| T_1 X_{B(\infty,\delta)} \right\|_{\mathcal{H}^\infty} \leq |T_1(j\infty)| + \left\| (T_1 - T_1(j\infty)) X_{B(\infty,\delta)} \right\|_{\mathcal{H}^\infty} \leq |T_1(j\infty)| + \epsilon.$$

It also follows that

$$\left\| T_1 X_{B(\omega_k,\delta)} \right\|_{\mathcal{H}^\infty} \leq |T_1(j\omega_k)| + \left\| (T_1 - T_1(j\omega_k)) X_{B(\omega_k,\delta)} \right\|_{\mathcal{H}^\infty} \leq |T_1(j\omega_k)| + \epsilon$$

for each k. Combining the above, then gives us

$$\| T_1 X_{S_\delta} \|_{\mathcal{H}^\infty} \leq \sup_{z \in \{\infty, \{\omega_k\}_{k=0}^{l}\}} \left\| T_1 X_{B(z,\delta)} \right\|_{\mathcal{H}^\infty} \leq \sup_{z \in \{\infty, \{\omega_k\}_{k=0}^{l}\}} |T_1(jz)| + \epsilon.$$

---

[3] The point $-\infty$ is treated analogously.

[4] The points $-\omega_k$ are treated analogously.

This, however, is equivalent to

$$\|T_1 X_{S_\delta}\|_{\mathcal{H}^\infty} \leq \max\{|T_1(j\infty)|, \max_k |T_1(j\omega_k)|, \left\|\Gamma_{T_1 T_{2_i}^*}\right\|\} + \epsilon.$$

This gives us an upper bound for T_1 on S_δ. We emphasize that this upper bound was obtained by exploiting the *continuity of* T_1 and the fact that T_{2_o} *only has a finite number of imaginary zeros.*

(c) Inequality on S_δ.

From proposition 5.2.3, we know that there exists $M_1 \stackrel{\text{def}}{=} M_1(\epsilon, T_1) \in Z_+$ (independent of ω) such that

$$|Y_m|_{[s=j\omega]} \leq \max\{|T_1(j\omega)|, \left\|\Gamma_{T_1 T_{2_i}^*}\right\|\} + \epsilon$$

for all $m \geq M_1$. In obtaining this, we have exploited the *magnitude and phase properties* of f_m.

Since M_1 is independent of ω, the inequality holds (almost) everywhere in S_δ. Given this, we obtain $\|Y_m X_{S_\delta}\|_{\mathcal{H}^\infty} \leq \max\{\|T_1 X_{S_\delta}\|_{\mathcal{H}^\infty}, \left\|\Gamma_{T_1 T_{2_i}^*}\right\|\} + \epsilon$ for all $m \geq M_1$.

Combining this inequality, with the upper bound obtained above for T_1 on S_δ, gives $\|Y_m X_{S_\delta}\|_{\mathcal{H}^\infty} \leq \max\{|T_1(j\infty)|, \max_k |T_1(j\omega_k)|, \left\|\Gamma_{T_1 T_{2_i}^*}\right\|\} + 2\epsilon$ for all $m \geq M_1$.

From lemma 5.2.1, we have $\max\{|T_1(j\infty)|, \max_k |T_1(j\omega_k)|, \left\|\Gamma_{T_1 T_{2_i}^*}\right\|\} \leq \mu_o$. Given this, we then have

$$\|Y_m X_{S_\delta}\|_{\mathcal{H}^\infty} \leq \mu_o + 2\epsilon$$

for all $m \geq M_1$. This completes the analysis on S_δ. We now turn to R/S_δ.

Step 2: Analysis on R/S_δ.

We now consider the set $R/S_\delta \stackrel{\text{def}}{=} R/B(\infty, \delta) \cup \cup_{k=0}^{l} B(\omega_k, \delta)$. On this set, for each $m \in Z_+$ we have

$$\left\|Y_m X_{R/S_\delta}\right\|_{\mathcal{H}^\infty} = \left\|(Y f_m + T_1(1 - f_m))X_{R/S_\delta}\right\|_{\mathcal{H}^\infty} \leq \|Y\|_{\mathcal{H}^\infty} \|f_m\|_{\mathcal{H}^\infty} + \|T_1\|_{\mathcal{H}^\infty} \left\|(1 - f_m)X_{R/S_\delta}\right\|_{\mathcal{H}^\infty}.$$

Since $\|f_m\|_{\mathcal{H}^\infty} \leq 1$ for each $m \in Z_+$, by proposition 5.2.3, this becomes

$$\left\|Y_m X_{R/S_\delta}\right\|_{\mathcal{H}^\infty} \leq \|Y\|_{\mathcal{H}^\infty} + \|T_1\|_{\mathcal{H}^\infty} \left\|(1 - f_m)X_{R/S_\delta}\right\|_{\mathcal{H}^\infty}$$

for each $m \in Z_+$

By proposition 5.2.3, the second term can be made arbitrarily small ($\leq 2\epsilon$) by taking m sufficiently large. How large m needs to be, depends on ϵ, on $\|T_1\|_{\mathcal{H}^\infty}$, on $\delta(\epsilon, T_1)$, and on $\Omega(\epsilon, T_1)$. Consequently, there exists $M_2 \stackrel{\text{def}}{=} M_2(\epsilon, T_1) \in Z_+$ such that

$$\left\|Y_m X_{R/S_\delta}\right\|_{\mathcal{H}^\infty} \leq \|Y\|_{\mathcal{H}^\infty} + 2\epsilon$$

for all $m \geq M_2$. In obtaining this, we have exploited the *magnitude and compact convergence properties* of f_m.

From proposition 5.2.1, $\|Y\|_{\mathcal{H}^\infty} = \left\|\Gamma_{T_1 T_{2_i}^*}\right\|$. From lemma 5.2.1 we have $\left\|\Gamma_{T_1 T_{2_i}^*}\right\| \leq \mu_o$. Given this, we then have

$$\left\|Y_m X_{R/S_\delta}\right\|_{\mathcal{H}^\infty} \leq \mu_o + 2\epsilon$$

for all $m \geq M_2$. This completes the analysis on R/S_δ.

Step 3: Combining steps (1) and (2).

Combining the results of (1) and (2) assures the existence of an integer $M \stackrel{\text{def}}{=} M(\epsilon, T_1) \stackrel{\text{def}}{=} \max\{M_1, M_2\} \in Z_+$, such that

$$\|Y_m\|_{\mathcal{H}^\infty} \leq \mu_o + 2\epsilon$$

for all $m \geq M$. This completes the proof. ∎

Comment 5.2.7 (Sifting Property of f_m)
Upon inspection of the above proof, we see that

$$\|T_1 + (Y - T_1)f_m\|_{\mathcal{H}^\infty} \leq \max\{\ |T_1(j\infty)|,\ \max_k |T_1(j\omega_k)|,\ \|Y\|_{\mathcal{H}^\infty}\ \} + 2\epsilon$$

for m sufficiently large. This *sifting property* of f_m, we emphasize, is mainly attributed to the magnitude and phase properties of f_m. Of course, the continuity of T_1 is also important.

∎

As we shall see, theorem 5.2.1 above and theorem 5.2.2 below are at the heart of the $\mathcal{H}^\infty$ results presented in the thesis. Together they will show how one can modify the Z_o to the "inner problem" considered in proposition 5.2.1, in order to construct nearly-optimal solutions to the $\mathcal{H}^\infty$ *Model Matching Problem* defined in definition 5.2.1.

We now construct the sequence $\{g_n\}_{n=1}^\infty$. To do so, we exploit the fact the zero structure of T_{2_o} on the extended imaginary axis.

Proposition 5.2.4 (Inverting Sequence for T_{2_o})
There exists a sequence $\{g_n\}_{n=1}^\infty \subset \mathcal{H}^\infty$ of outer functions which possess the following properties:

(1) $\{g_n\}_{n=1}^\infty$ is uniformly bounded in $\mathcal{H}^\infty$.
(2) $T_{2_o}^{-1} g_n$, $T_{2_o} g_n^{-1} \in \mathcal{H}^\infty \cap C_e$ for each $n \in Z_+$.
(3) g_n uniformly approximates unity everywhere except on open neighborhoods of the points $\{j\omega_k\}_{k=0}^l$; i.e. (see definition 2.10.4) for each $\delta > 0$,

$$\lim_{n\to\infty} \left\|(1 - g_n)X_{R/\cup_{k=0}^l B(\omega_k,\delta)}\right\|_{\mathcal{H}^\infty} = 0.$$

(4) $g_m^{-1} g_n \in \mathcal{H}^\infty \cap C_e$ for each $m, n \in Z_+$.

(5) If $m_k > 0$, then $g_n(\omega_k) = 0$ for each $n \in Z_+$.

We refer to $\{g_n\}_{n=1}^\infty$ as an inverting sequence for T_{2_o}.

∎

Proof The sequence defined by the function

$$g_n(s) \stackrel{\text{def}}{=} \left(\frac{s}{s+\frac{1}{n}}\right)^{m_0} \prod_{k=1}^{l-1} \left(\frac{s^2+\omega_k^2}{s^2+2\frac{1}{n}s+\omega_k^2+\frac{1}{n}^2}\right)^{m_k} \left(\frac{n}{s+n}\right)^{m_l}$$

satisfies the conditions of the proposition. We emphasize that g_n is, in general, irrational since the m_k, in general, are fractions. We also note that the sequence $\{g_n\}_{n=1}^\infty$ constructed above is the

same inverting sequence that we would use if T_{2_o} were real-rational. Finally, it should be noted that

$$g_n(s) = F(s)\, F^{-1}(s + \frac{1}{n})\, G_n(s).$$

This completes the construction. ∎

Example 5.2.1 (Sample Functions: T_{2o} and g_n)
The following indicate that proposition 5.2.4 covers a large class of outer functions T_{2_o}.

(a) If $T_{2_o} = \frac{1}{s+1}$ choose $g_n = \frac{n}{s+n}$.

(b) If $T_{2_o} = \frac{s}{s+2}$ choose $g_n = \frac{s}{s+\frac{1}{n}}$.

(c) If $T_{2_o} = \sqrt[5]{\frac{s}{s+3}}$ choose $g_n = \sqrt[5]{\frac{s}{s+\frac{1}{n}}}$.

(d) More complicated outer functions may require the use of [24, pp. 85; theorem 7.4] to construct g_n. The results in [17] may also prove useful.

∎

Combining the previous theorem and propositions leads us to the following key theorem.

Theorem 5.2.2 (Near-Optimal Irrational Solution)
Let

$$Q_{m,n} \stackrel{\text{def}}{=} Q_o f_m g_n$$

where $Q_o \stackrel{\text{def}}{=} {T_{2_o}}^{-1} Z_o$. *Then,* $Q_{m,n} \in \mathcal{H}_0^\infty$ *for each* $m, n \in Z_+$. *Also, for each* $m, n \in Z_+$, *we have* $Q_{m,n}(j\omega_k) = 0$ *for each* $k = 0, 1, \ldots, l$. *Moreover, there exists* $M \stackrel{\text{def}}{=} M(\epsilon, T_1) \in Z_+$ *and* $N \stackrel{\text{def}}{=} N(\epsilon, M) \in Z_+$ *such that*

$$\mu_o \le \|T_1 - T_2 Q_{M,n}\|_{\mathcal{H}^\infty} \le \mu_o + 3\epsilon$$

for all $n \ge N$.

Finally, from the integer valued functions defined by $M(\epsilon, T_1)$ *and* $N(\epsilon, M)$, *it follows that we can construct a sequence* $\{Q_m\}_{m=1}^{\infty}$ *such that*

(1) $Q_m \in \mathcal{H}_0^\infty$ *for each* $m \in Z_+$.

(2) If $m_k > 0$, *then* $Q_m(j\omega_k) = 0$ *for each* $m \in Z_+$.

(3) $\mu_o \le \|T_1 - T_2 Q_m\|_{\mathcal{H}^\infty} \le \mu_o + 3\epsilon$ *for all* $m \ge M(\epsilon, T_1)$.

∎

Proof
The proof will proceed in three steps.

Step 1: Admissibility of $Q_{m,n}$.

We have

$$Q_{m,n} \stackrel{\text{def}}{=} T_{2_o}{}^{-1} Z_o f_m g_n.$$

By proposition 5.2.3, $f_m \in \mathcal{H}_0^\infty$ for each $m \in Z_+$. By proposition 5.2.4, $T_{2_o}{}^{-1} g_n \in \mathcal{H}^\infty$ for each $n \in Z_+$. Combining these gives us that $Q_{m,n} \in \mathcal{H}_0^\infty$ for each $m, n \in Z_+$.

By proposition 5.2.3, $f_m(j\omega_k) = 0$ for each $m \in Z_+$ and for each $k = 1, 2, \ldots, l$. Since $T_{2_o}{}^{-1} g_n \in \mathcal{H}^\infty$ for each $n \in Z_+$, it follows that it cannot have poles at the points $j\omega_k$ for any n. This implies that it cannot cancel the imaginary zeros of f_m for any m. Consequently, $Q_{m,n}(j\omega_k) = 0$ for each $m, n \in Z_+$ and for each $k = 0, 1, \ldots, l$.

Step 2: Near-Optimality of $Q_{M,n}$.

(a) Main Inequality

To prove the rest of the theorem, we use the following inequality:

$$\|T_1 - T_2 Q_{m,n}\|_{\mathcal{H}^\infty} = \|T_1 - T_{2_i} Z_o f_m g_n\|_{\mathcal{H}^\infty} \le \|T_1 - T_{2_i} Z_o f_m\|_{\mathcal{H}^\infty} + \|T_{2_i} Z_o f_m (1 - g_n)\|_{\mathcal{H}^\infty}.$$

(b) Use of f_m to Modify Inner Solution.

From theorem 5.2.1, $\|T_1 - T_{2_i} Z_o f_m\|_{\mathcal{H}^\infty} \le \mu_o + 2\epsilon$ for all $m \ge M \stackrel{\text{def}}{=} M(\epsilon, T_1)$. Let $m = M$. This then gives

$$\|T_1 - T_2 Q_{M,n}\|_{\mathcal{H}^\infty} \le \mu_o + 2\epsilon + \|T_{2_i} Z_o f_M (1 - g_n)\|_{\mathcal{H}^\infty}.$$

(c) Use of g_n to Complete Inversion of T_{2_o}.

From proposition 5.2.3, we have that $f_M \in \mathcal{H}_0^\infty$ and is zero and continuous at each point $j\omega_k$. From proposition 5.2.4, g_n uniformly approximates unity everywhere except on open neighborhoods of the points $\{j\omega_k\}_{k=0}^{l}$; i.e. (see definition 2.10.4) for each $\delta > 0$,

$$\lim_{n \to \infty} \left\| (1 - g_n) X_{R / \cup_{k=0}^{l} B(\omega_k, \delta)} \right\|_{\mathcal{H}^\infty} = 0.$$

From proposition 5.2.4, the g_n are uniformly bounded. Given this, it follows from lemma 2.10.1 that we can make the last term in the above inequality arbitrarily small by taking n sufficiently large. More precisely, there exists $N \stackrel{\text{def}}{=} N(\epsilon, M) \in Z_+$ such that

$$\|T_{2_i} Z_o f_M (1 - g_n)\|_{\mathcal{H}^\infty} \le \epsilon$$

for all $n \ge N$. This implies that

$$\|T_1 - T_2 Q_{M,n}\|_{\mathcal{H}^\infty} \le \mu_o + 3\epsilon.$$

for all $n \ge N$. This proves the first upper inequality. The lower inequality follows from the definition of μ_o and the fact that $\{Q_{M,n}\}_{n=N}^{\infty} \subset \mathcal{H}_0^\infty$.

Step 3: Construction of Q_m.

The construction of the sequence $\{Q_m\}_{m=1}^{\infty}$ follows from the above. This completes the proof. ∎

Theorems 5.2.1 and 5.2.2 show that under the very weak assumption 5.2.1, we can modify any solution Z_o of the "inner problem" to find a nearly-optimal Q-parameter for the $\mathcal{H}^\infty$ *Model Matching Problem* defined in definition 5.2.1. Moreover, in the proof of theorems 5.2.1 and 5.2.2 we have indicated how such a Q-parameter can be constructed. The proofs show that to construct a near-optimal solution one needs to *invert* T_{2_o} *away from its zeros in the closed right half plane.* The following example serves to further illustrate the construction.

Example 5.2.2 (Sample Functions: T_{2o}, f_m, and g_n)
Suppose that $T_{2_o} = \left(\frac{1}{s+1}\right)\left(\frac{s^2+1}{s^2+5s+6}\right)\left(\frac{s}{s+4}\right)^{\frac{1}{2}}$.

The sequence defined by the function $f_m = \left(\frac{1000}{s+1000}\right)^{\frac{1}{m}}\left(\frac{s-j1}{s-j1+.0001}\right)^{\frac{1}{m}}\left(\frac{s+j1}{s+j1+.0001}\right)^{\frac{1}{m}}\left(\frac{s}{s+.0001}\right)^{\frac{1}{m}}$ satisfies the properties in proposition 5.2.3.

The sequence defined by the function $g_n = \left(\frac{n}{s+n}\right)\left(\frac{s-j1}{s-j1+\frac{1}{n}}\right)\left(\frac{s+j1}{s+j1+\frac{1}{n}}\right)\left(\frac{s}{s+\frac{1}{n}}\right)^{\frac{1}{2}}$ satisfies the conditions in proposition 5.2.4.

Note:
The sequences $\{f_m\}_{m=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ depend only on the zero structure of T_{2_o}. They do not not depend on T_{2_i}.

∎

Although the construction presented in theorem 5.2.2 has been obtained under the weak assumption 5.2.1, the ideas cover a much broader class of T_1 and T_2. The following comment is met to illustrate this point.

Comment 5.2.8 (Form of Construction, Generality, Main Ideas)
In summary, it has been shown that if there exists functions F and G_a which satisfy the following conditions

(1) $T_{2_o}^{-1}(\cdot)\, F(\cdot)\, F^{-1}(\cdot + \frac{1}{a})\, G_a(\cdot) \in \mathcal{H}^\infty$ for each $a \in Z_+$,

(2) $\lim_{n\to\infty} \left\| F(\cdot)\, F^{-1}(\cdot + \frac{1}{n})\, G_m(\cdot) \right\|_{\mathcal{H}^\infty} \leq 1$,

(3) $\lim_{n\to\infty} \left\| \theta_{F(\cdot)\, F^{-1}(\cdot+\frac{1}{n})\, G_n(\cdot)} \right\|_{\mathcal{L}^\infty} \leq B$ for some $B \in R_+$,

(4) The sequence $\{F(\cdot)\, F^{-1}(\cdot + \frac{1}{n})\, G_n(\cdot)\}_{n=1}^{\infty}$ approximates unity on all compact frequency intervals not including an imaginary zero of T_{2_o},

then a near-optimal solution to the $\mathcal{H}^\infty$ *Model Matching Problem* can be constructed from the double sequence defined by the following function

$$Q_{m,n} = T_{2_o}^{-1}\, Z_o\, f_m\, g_n$$

where

$$f_m = [\, F(s)\, F^{-1}(s + \frac{1}{m})\, G_m(s)\, \frac{m}{s+m}\,]^{\frac{1}{m}}$$

and

$$g_m = [\, F(s)\, F^{-1}(s + \frac{1}{n})\, G_n(s)\,].$$

Such functions F and G_a were shown to exist under the weak assumption 5.2.1.

The ideas which permitted the above construction shall be used extensively throughout the thesis. We thus point out the "main ideas".

In the construction we modify the "inner solution" Z_o by "rolling it off" with f_m as in proposition 5.2.3. This really is the main step. It can be done primarily because of the special phase property of the irrational roll-off function f_m; i.e. the phase of f_m can be made arbitrarily small, uniformly in frequency, by taking m sufficiently large; i.e.

$$\lim_{m\to\infty} \|\theta_{f_m}\|_{\mathcal{L}^\infty} = 0.$$

We then choose $m = M$ sufficiently large. How large M must be chosen depends on ϵ and T_1. Finally, we show that given M, N can be chosen sufficently large so that

$$Q_{M,N} \stackrel{\text{def}}{=} T_{2_o}{}^{-1} Z_o f_M g_{N(M)}$$

is near-optimal. How large N must be depends on ϵ and M. Here the main role of g_n is to assure the strict propriety of $Q_{M,N}$ and to assure that it has no poles on the imaginary axis. It allows us to invert T_{2_o} in a stable manner.

■

Thus far we have only addressed the problem of constructing near-optimal solutions for the $\mathcal{H}^\infty$ *Model Matching Problem* defined in definition 5.2.1. We now address the computation of the quantity μ_o.

5.3 Computation of μ_o

Often it is necessary to determine μ_o. The following proposition indicates how one might, in priciple, compute the value μ_o. All assumptions made in the previous section are assumed to hold here.

Proposition 5.3.1 *Each of the following equal* μ_o*:*

$$\inf_{Q\in\mathcal{H}_0^\infty} \|T_1 - T_2 Q\|_{\mathcal{H}^\infty} \tag{1}$$

$$\inf_{\substack{Q\in\mathcal{H}_0^\infty \\ \{Q(j\omega_k)=0;\ k=0,1,\ldots,l\}}} \|T_1 - T_{2_i} Q\|_{\mathcal{H}^\infty} \tag{2}$$

$$\max\{\ |T_1(j\infty)|,\ \max_k |T_1(j\omega_k)|,\ \left\|\Gamma_{T_1 T_{2_i}^*}\right\|\ \} \tag{3}$$

■

Proof The proof shall proceed as follows: (a) 2=3 (b) 1=3.

(a) (2=3)

We claim that the key to proving proposition 5.3.1 is showing that (2=3); i.e.

$$\inf_{\substack{Q\in\mathcal{H}_0^\infty \\ \{Q(j\omega_k)=0;\ k=0,1,\cdots,l\}}} \|T_1 - T_{2_i} Q\|_{\mathcal{H}^\infty} = \mu$$

where

$$\mu \stackrel{\text{def}}{=} \max\{|T_1(j\infty)|, \max_k |T_1(j\omega_k)|, \left\|\Gamma_{T_1 T_{2_i}^*}\right\|\}.$$

We know from proposition 5.2.1 that $\inf_{Z\in\mathcal{H}^\infty} \|T_1 - T_{2_i}Z\|_{\mathcal{H}^\infty} = \left\|\Gamma_{T_1T_{2_i}^*}\right\|$. Since the set of $Q \in \mathcal{H}_0^\infty$ which satisfy $Q(j\omega_k) = 0$ for each k also lie in $\mathcal{H}^\infty$, we have

$$\left\|\Gamma_{T_1T_{2_i}^*}\right\| = \inf_{Z\in\mathcal{H}^\infty} \|T_1 - T_{2_i}Z\|_{\mathcal{H}^\infty} \le \inf_{\substack{Q\in\mathcal{H}_0^\infty \\ \{Q(j\omega_k)=0;\ k=0,1,\ldots,l\}}} \|T_1 - T_{2_i}Q\|_{\mathcal{H}^\infty}.$$

From proposition 2.7.2 we have $|T_1(j\infty)| \le \|T_1 - T_{2_i}Q\|_{\mathcal{H}^\infty}$ for each $Q \in \mathcal{H}_0^\infty$, including those which satisfy $Q(j\omega_k) = 0$ for each k. Consequently,

$$|T_1(j\infty)| \le \inf_{\substack{Q\in\mathcal{H}_0^\infty \\ \{Q(j\omega_k)=0;\ k=0,1,\ldots,l\}}} \|T_1 - T_{2_i}Q\|_{\mathcal{H}^\infty}.$$

For each $Q \in \mathcal{H}_0^\infty$ which satisfies $Q(j\omega_k) = 0$ for each k, proposition 2.7.2 gives us $|T_1(j\omega_k)| \le \|T_1 - T_{2_i}Q\|_{\mathcal{H}^\infty}$. Consequently,

$$\max_k |T_1(j\omega_k)| \le \inf_{\substack{Q\in\mathcal{H}_0^\infty \\ \{Q(j\omega_k)=0;\ k=0,\ldots,l\}}} \|T_1 - T_{2_i}Q\|_{\mathcal{H}^\infty}.$$

Combining the above then gives

$$\mu \stackrel{\text{def}}{=} \max\{|T_1(j\infty)|, \max_k |T_1(j\omega_k)|, \|Y\|_{\mathcal{H}^\infty}\} \le \inf_{\substack{Q\in\mathcal{H}_0^\infty \\ \{Q(j\omega_k)=0;\ k=0,\ldots,l\}}} \|T_1 - T_{2_i}Q\|_{\mathcal{H}^\infty}.$$

To show that (2=3), we thus only need to prove the converse inequality. To prove the converse inequality, it suffices to construct a sequence $\{Z_m\}_{m=1}^\infty \subset \mathcal{H}_0^\infty$ such that $Z_m(j\omega_k) = 0$ for each $k = 0, 1, \ldots, l$, and such that

$$\|T_1 - T_{2_i}Z_m\|_{\mathcal{H}^\infty} \le \max\{|T_1(j\infty)|, \max_k |T_1(j\omega_k)|, \left\|\Gamma_{T_1T_{2_i}^*}\right\|\} + 3\epsilon$$

for m sufficiently large. The existence of such a sequence, however, is guaranteed by theorem 5.2.1; just choose $Z_m = Z_o f_m$. This proves the converse inequality and hence that (2=3).

(b) (1=3)

We now prove that (1=3); i.e.

$$\mu_o \stackrel{\text{def}}{=} \inf_{Q\in\mathcal{H}_0^\infty} \|T_1 - T_2Q\|_{\mathcal{H}^\infty} = \mu,$$

where again

$$\mu \stackrel{\text{def}}{=} \max\{|T_1(j\infty)|, \max_k |T_1(j\omega_k)|, \left\|\Gamma_{T_1T_{2_i}^*}\right\|\}.$$

From lemma 5.2.1, we have that

$$\mu \le \mu_o.$$

To prove that (1=3) we need to prove the converse inequality. To do so it suffices to construct a sequence $\{Q_m\}_{m=1}^\infty \subset \mathcal{H}_0^\infty$ such that

$$\|T_1 - T_2Q_m\|_{\mathcal{H}^\infty} \le \max\{|T_1(j\infty)|, \max_k |T_1(j\omega_k)|, \left\|\Gamma_{T_1T_{2_i}^*}\right\|\} + 3\epsilon$$

for m sufficiently large. The existence of such a sequence, however, is guaranteed by theorem 5.2.2; just choose $Q_m = T_{2_o}{}^{-1} Z_o f_m g_n$ where $m = M(\epsilon, T_1)$ and $n = N(\epsilon, T_1)$ as defined in the theorem.

This proves the converse inequality and hence that (1=3). ∎

Proposition 5.3.1 shows that

$$\mu_o = \max\{|T_1(j\infty)|, \max_k |T_1(j\omega_k)|, \left\|\Gamma_{T_1 T_{2_i}^*}\right\|\}.$$

Given this, we have the following corollary.

Corollary 5.3.1 (Spectral Result)
If

$$\max\{|T_1(j\infty)|, \max_k |T_1(j\omega_k)|\} \leq \left\|\Gamma_{T_1 T_{2_i}^*}\right\|,$$

then

$$\mu_o = \left\|\Gamma_{T_1 T_{2_i}^*}\right\|.$$

∎

This result shows that if the assumption in corollary 5.3.1 is satisfied, then to compute μ_o all that we need do is compute $\left\|\Gamma_{T_1 T_{2_i}^*}\right\|$; the operator norm of the Hankel operator $\Gamma_{T_1 T_{2_i}^*}$. This, in general, involves solving an infinite-dimensional eigenvalue/eigenfunction problem [61]. This computation can be particularly difficult when the operator is not compact. In the sequel we will show how this computation, in many cases (including the non-compact case), may be carried out by solving a sequence of finite-dimensional eigenvalue/eigenvector problems instead.

The following lemma is presented to show how one might construct T_1 so that the assumption of corollary 5.3.1 is satisfied.

Lemma 5.3.1 (Construction of T_1)
Suppose that $T_1^{-1} \in \mathcal{H}^B$ and satisfies

$$\max\{|T_1(j\infty)|, \max_k |T_1(j\omega_k)|\} = \inf_{\omega \in R_e} |T_1(j\omega)|.$$

Also, let z be any point in the extended open right half plane, such that $T_{2_i}(z) = 0$. We then have

$$\max\{|T_1(j\infty)|, \max_k |T_1(j\omega_k)|\} \leq \left\|\Gamma_{T_1 T_{2_i}^*}\right\|.$$

∎

Proof From proposition 5.2.1, there exists $Z_o \in \mathcal{H}^\infty$ such that

$$\left\|\Gamma_{T_1 T_{2_i}^*}\right\| = \|T_1 - T_{2_i} Z_0\|_{\mathcal{H}^\infty}.$$

From proposition 2.7.2, we have

$$\geq |T_1(z)|$$

Since $T_1^{-1} \in \mathcal{H}^B$, the minimum modulus theorem [50, pp. 159] can be applied to get

$$\geq \inf_{\omega \in R_e} |T_1(j\omega)|.$$

The result then follows since the right hand side equals $\max\{|T_1(j\infty)|, \max_k |T_1(j\omega_k)|\}$ by construction. ∎

This lemma can be applied to $T_{2_i} = e^{-s}$, for example.

5.4 Computation of Hankel Norm

The difficulty in computing μ_o can be attributed to the difficulty in computing the Hankel norm $\left\|\Gamma_{T_1 T_{2_i}^*}\right\|$. This computation typically requires that one solve an infinite dimensional eigenvalue/eigenfunction problem [14, pp. 28–31], [61], [63, pp. 308]. In this section, it is shown how the computation can be carried out by solving a sequence of finite dimensional eigenvalue/eigenvector problems.

Throughout this section we assume that $\{T_{1_n}\}_{n=1}^{\infty}, \{T_{2_{n_i}}\}_{n=1}^{\infty} \subset R\mathcal{H}^\infty$, where $T_{2_{n_i}}$ is inner in $\mathcal{H}^\infty$ for all $n \in Z_+$. We begin with the following proposition. Loosely speaking, it shows that $\left\|\Gamma_{T_1 T_{2_i}^*}\right\|$ is lower-semicontinuous in T_{2_i} in the compact topology on $\mathcal{H}^\infty$.

Proposition 5.4.1 ("Lower-semicontinuity" Hankel Result)
Suppose that

$$\lim_{n\to\infty} \|T_{1_n} - T_1\|_{\mathcal{H}^\infty} = 0$$

and

$$\lim_{n\to\infty} \left\|(T_{2_{n_i}} - T_{2_i})X_{[-\Omega,\Omega]}\right\|_{\mathcal{H}^\infty} = 0$$

for each $\Omega \in R_+$, however large. It then follows that

$$\left\|\Gamma_{T_1 T_{2_i}^*}\right\| \le \lim_{n\to\infty} \left\|\Gamma_{T_{1_n} T_{2_{n_i}}^*}\right\|.$$

■

Proof Let $v \in \mathcal{L}^2$ be such that $\|v\|_{\mathcal{L}^2} = 1$ and

$$\left\|\Gamma_{T_1 T_{2_i}^*}\right\| \le \left\|\Gamma_{T_1 T_{2_i}^*} v\right\|_{\mathcal{L}^2} + \epsilon.$$

Given this, it follows that

$$\left\|\Gamma_{T_1 T_{2_i}^*}\right\| \le \left\|\Gamma_{T_1 (T_{2_i} - T_{2_{n_i}} + T_{2_{n_i}})^*} v\right\|_{\mathcal{L}^2} + \epsilon \le \left\|\Gamma_{T_1 (T_{2_i} - T_{2_{n_i}})^*} v\right\|_{\mathcal{L}^2} + \left\|\Gamma_{T_1 T_{2_{n_i}}^*} v\right\|_{\mathcal{L}^2} + \epsilon$$

$$\le \left\|T_1 (T_{2_i} - T_{2_{n_i}})^* v\right\|_{\mathcal{L}^2} + \left\|\Gamma_{T_1 T_{2_{n_i}}^*} v\right\|_{\mathcal{L}^2} + \epsilon.$$

Since v is not necessarily a "maximal vector" for $\Gamma_{T_1 T_{2_{n_i}}^*}$, it follows that

$$\left\|\Gamma_{T_1 T_{2_i}^*}\right\| \le \left\|T_1 (T_{2_i} - T_{2_{n_i}})^* v\right\|_{\mathcal{L}^2} + \left\|\Gamma_{T_1 T_{2_{n_i}}^*}\right\| + \epsilon$$

From lemma 2.10.2 and our assumption on $T_{2_{n_i}}$, it follows that the first term can be made arbitrarily small by taking n sufficiently large. Consequently, there exists $N \stackrel{\text{def}}{=} N(\epsilon, T_1, v)$ such that

$$\left\|\Gamma_{T_1 T_{2_i}^*}\right\| \le \left\|\Gamma_{T_1 T_{2_{n_i}}^*}\right\| + 2\epsilon$$

for all $n \ge N$. We also have

$$\left\|\Gamma_{T_1 T_{2_{n_i}}^*}\right\| \le \left\|\Gamma_{T_{1_n} T_{2_{n_i}}^*}\right\| + \left\|\Gamma_{(T_1 - T_{1_n}) T_{2_{n_i}}^*}\right\| \le \left\|\Gamma_{T_{1_n} T_{2_{n_i}}^*}\right\| + \|T_1 - T_{1_n}\|_{\mathcal{H}^\infty}.$$

Combining this inequality with the previous, gives the result. ■

Since $T_1 \in \mathcal{C}_e$, we know from proposition 2.10.1 that there exists a sequence of $R\mathcal{H}^\infty$ functions $\{T_{1_n}\}_{n=1}^{\infty}$ which uniformly approximate T_1. Given this proposition, we have the following theorem which gives insight into the computation of $\left\|\Gamma_{T_1 T_{2_i}^*}\right\|$.

Theorem 5.4.1 (Spectral Result: Computation of Hankel Norms)
Given that

$$\lim_{n\to\infty} \|T_{1_n} - T_1\|_{\mathcal{H}^\infty} = 0,$$

each of the following imply that $\lim_{n\to\infty} \left\|\Gamma_{T_1 T^*_{2_{n_i}}}\right\| = \left\|\Gamma_{T_{1_n} T^*_{2_{n_i}}}\right\|$.

$$\lim_{n\to\infty} \left\|T_{2_{n_i}} - T_{2_i}\right\|_{\mathcal{H}^\infty} = 0 \tag{1}$$

$$\lim_{n\to\infty} \left\|T_1(T_{2_{n_i}} - T_{2_i})\right\|_{\mathcal{H}^\infty} = 0 \tag{2}$$

$$T_1(j\infty) = 0; \;\lim_{n\to\infty} \left\|(T_{2_{n_i}} - T_{2_i})X_{[-\Omega,\Omega]}\right\|_{\mathcal{H}^\infty} = 0 \tag{3}$$

$$|T_1(j\infty)| \leq \left\|\Gamma_{T_1 T^*_{2_i}}\right\|; \;\lim_{n\to\infty} \left\|(T_{2_{n_i}} - T_{2_i})X_{[-\Omega,\Omega]}\right\|_{\mathcal{H}^\infty} = 0 \tag{4}$$

for each $\Omega \in R_+$, *however large.*

■

Proof The proof of $(1) - (3)$ follows from the following inequality

$$\left|\,\left\|\Gamma_{T_{1_n} T^*_{2_{n_i}}}\right\| - \left\|\Gamma_{T_1 T^*_{2_i}}\right\|\,\right| \leq \left\|\Gamma_{T_{1_n} T^*_{2_{n_i}} - T_1 T^*_{2_i}}\right\|$$

$$\leq \left\|\Gamma_{(T_{1_n} - T_1)T^*_{2_{n_i}} + T_1(T_{2_{n_i}} - T_{2_i})^*}\right\| \leq \|T_{1_n} - T_1\|_{\mathcal{H}^\infty} + \left\|T_1(T_{2_{n_i}} - T_{2_i})\right\|_{\mathcal{H}^\infty}.$$

To prove (4), we invoke proposition 5.4.1 to obtain

$$\left\|\Gamma_{T_1 T^*_{2_i}}\right\| \leq \lim_{n\to\infty} \left\|\Gamma_{T_{1_n} T^*_{2_{n_i}}}\right\|.$$

This proves one direction. To prove the other direction we proceed as follows. Let $Q_o \in \mathcal{H}^\infty_0$ be such that

$$\|T_1 - T_{2_i} Q_o\|_{\mathcal{H}^\infty} \leq \max\{\, |T_1(j\infty)|, \left\|\Gamma_{T_1 T^*_{2_i}}\right\| \,\} + \epsilon.$$

The existence of such a Q_o is guaranteed by theorem 5.2.1. Given this, we have

$$\left\|\Gamma_{T_{1_n} T^*_{2_{n_i}}}\right\| = \inf_{Z\in\mathcal{H}^\infty} \left\|T_{1_n} - T_{2_{n_i}} Z\right\|_{\mathcal{H}^\infty} \leq \left\|T_{1_n} - T_{2_{n_i}} Q_o\right\|_{\mathcal{H}^\infty}$$

$$\leq \|T_1 - T_{2_i} Q_o\|_{\mathcal{H}^\infty} + \|T_{1_n} - T_1\|_{\mathcal{H}^\infty} + \left\|(T_{2_i} - T_{2_{n_i}})Q_o\right\|_{\mathcal{H}^\infty}$$

$$\leq \max\{\, |T_1(j\infty)|, \left\|\Gamma_{T_1 T^*_{2_i}}\right\| \,\} + \epsilon + \|T_{1_n} - T_1\|_{\mathcal{H}^\infty} + \left\|(T_{2_i} - T_{2_{n_i}})Q_o\right\|_{\mathcal{H}^\infty}.$$

By assumption, $|T_1(j\infty)| \leq \left\|\Gamma_{T_1 T^*_{2_i}}\right\|$. Hence, the above inequality becomes

$$\left\|\Gamma_{T_1 T^*_{2_{n_i}}}\right\| \leq \left\|\Gamma_{T_1 T^*_{2_i}}\right\| + \epsilon + \|T_{1_n} - T_1\|_{\mathcal{H}^\infty} + \left\|(T_{2_i} - T_{2_{n_i}})Q_o\right\|_{\mathcal{H}^\infty}.$$

The second norm can be made arbitrarily small by taking n sufficiently large. This follows by our assumption that T_{1_n} approximates T_1 uniformly. By lemma 2.10.1, the last term can be

made arbitrarily small by taking n sufficiently large. This is because $T_{2_{n_i}}$ is uniformly bounded, it approximates T_{2_i} on compact frequency intervals, and Q_o rolls-off. Consequently, there exists $N \stackrel{\text{def}}{=} N(\epsilon, Q_o) \in Z_+$ such that

$$\left\| \Gamma_{T_1 T^*_{2_{n_i}}} \right\| \leq \left\| \Gamma_{T_1 T^*_{2_i}} \right\| + 3\epsilon$$

for all $n \geq N$. This proves the other direction and completes the proof. ∎

Comment 5.4.1 (Applicability, Spectral Implications)
We note that the criterion in (1) is a continuity statement. It implies that $\left\| \Gamma_{T_1 T^*_{2_i}} \right\|$ is continuous in T_{2_i} when T_{1_n} uniformly approximates T_1. Since $R\mathcal{H}^\infty$ is not dense in $\mathcal{H}^\infty$, the criterion in (1) is usually not applicable; e.g. $T_{2_i} = e^{-s}$ cannot be approximated uniformly by $R\mathcal{H}^\infty$ functions.

The criteria in (2) and (3) are helpful only when T_1 rolls-off.

Unlike the other criterion, the criterion in (4) is usually applicable. It applies to cases in which the Hankel operator $\Gamma_{T_1 T^*_{2_{n_i}}}$ is non-compact; e.g. $T_1 \in R\mathcal{H}^\infty$ proper and $T_{2_i} = e^{-s}$ (cf. proposition 2.9.2). In such cases, the non-compact Hankel operator cannot be approximated by finite rank operators. Moreover, it is not clear to us how one would approximate a non-compact Hankel operator[5]. It also applies in instances where T_{2_i} has an infinite number of poles and/or zeros; e.g. an infinite Blaschke product. To verify the inequality criterion in (4), lemma 5.3.1 may prove helpful. If ∞ is an essential singularity of T_{2_i}, then the inequality is automatically satisfied. This has been shown in [61].

Finally, the spectral implications of theorem 5.4.1 must be acknowlegded. More specifically, we reiterate that computing $\left\| \Gamma_{T_1 T^*_{2_i}} \right\|$ amounts to solving an infinite dimensional eigenvalue/eigenfunction problem [14, pp. 28–31], [61], [63, pp. 308]. Theorem 5.4.1 gives weak conditions under which such a computation can be carried out by solving a sequence of finite dimensional eigenvalue-eigenvector problems [23]. Since computing the above Hankel norm is crucial in many $\mathcal{H}^\infty$ design procedures, this theorem is extremely useful.

∎

5.5 Sequences of Finite Dimensional $\mathcal{H}^\infty$ Model Matching Problems

The $\mathcal{H}^\infty$ *Model Matching Problem*, in general, defines an infinite dimensional optimization problem. Obtaining a solution is often difficult and requires sophisticated mathematics [6]. It is thus natural to ask whether or not solving a sequence of appropriately formulated finite dimensional problems can yield desirable results. Solving a finite dimensional $\mathcal{H}^\infty$ *Model Matching Problem* requires little mathematical sophistication. Also, much software exists to support such an approach.

Let $\{T_{1_n}\}_{n=1}^\infty$ and $\{T_{2_n}\}_{n=1}^\infty$ be sequences of $R\mathcal{H}^\infty$ functions. Consider the finite dimensional $\mathcal{H}^\infty$ *Model Matching Problem* defined by the following optimization problem.

Definition 5.5.1

$$\mu_n \stackrel{\text{def}}{=} \inf_{Q \in R\mathcal{H}_0^\infty} \|T_{1_n} - T_{2_n} Q\|_{\mathcal{H}^\infty}$$

[5] Distribution theory may be helpful in answering this question.

[6] Computing Z_o may be difficult.

■

Our goal is to gain insight into the infinite dimensional problem defined in definition 5.2.1, by studying the finite dimensional model matching problem posed in definition 5.5.1. This, of course, is reasonable only if T_{1_n} and T_{2_n} approximate T_1 and T_2, in some sense. We thus make additional assumptions on the sequences $\{T_{1_n}\}_{n=1}^{\infty}$ and $\{T_{2_n}\}_{n=1}^{\infty}$.

Assumption 5.5.1 (T_{1_n} Approximates T_1 Uniformly)
Throughout the section, it shall be assumed that the sequence $\{T_{1_n}\}_{n=1}^{\infty}$ satisfies

$$\lim_{n\to\infty} \|T_{1_n} - T_1\|_{\mathcal{H}^\infty} = 0.$$

■

Since $T_1 \in \mathcal{C}_e$, we know from proposition 2.10.1 that it can be uniformly approximated by $R\mathcal{H}^\infty$ functions. Hence, this assumption is justified.

As with the $\mathcal{H}^\infty$ *Model Matching Problem*, the above finite dimensional model matching problem must be approached with an inversion mentality. Given this, it should be clear, for example, that we will need to invert the outer part $T_{2_{n_o}}$ of T_{2_n}, away from its imaginary zeros, in the closed right half plane. This statement has already been implemented for T_{2_o} in section 5.2. To do the same for $T_{2_{n_o}}$, we will need to construct T_2 in a clever way. It should be apparent, for example, that choosing $T_{2_{n_o}}$ so that it uniformly approximates T_{2_o} in $\mathcal{H}^\infty$, will not suffice. This is because $T_{2_{n_o}}$ and T_{2_o}, even for large n, may possess drastically different zero structures on the extended imaginary axis. We now show how to prevent this from occurring.

From proposition 5.2.4, we know that there exists a sequence $\{g_N\}_{N=1}^{\infty} \subset \mathcal{H}^\infty$ of outer functions such that

(1) $\{g_N\}_{N=1}^{\infty}$ is uniformly bounded in $\mathcal{H}^\infty$.
(2) $T_{2_o}^{-1} g_N$, $T_{2_o} g_N^{-1} \in \mathcal{H}^\infty \cap \mathcal{C}_e$ for each $N \in Z_+$.
(3) g_N uniformly approximates unity everywhere except on open neighborhoods of the points $\{j\omega_k\}_{k=0}^{l}$; i.e. (see definition 2.10.4) for each $\delta > 0$,

$$\lim_{N\to\infty} \left\|(1 - g_N)X_{R/\cup_{k=0}^{l} B(\omega_k,\delta)}\right\|_{\mathcal{H}^\infty} = 0.$$

(4) $g_M^{-1} g_N \in \mathcal{H}^\infty \cap \mathcal{C}_e$ for each $M, N \in Z_+$.

(5) If $m_k > 0$, then $g_N(\omega_k) = 0$ for each $N \in Z_+$.

The above properties were critical in "inverting" T_{2_o}. We referred to $\{g_N\}_{N=1}^{\infty}$ as an *inverting sequence* for T_{2_o}. The following proposition, we shall see, will guarantee the existence of such a sequence for $T_{2_{n_o}}$; i.e. provided that T_{2_n} is constructed appropriately.

Proposition 5.5.1 (Inverting Sequence for $T_{2_{n_o}}$)
Fix $N \in Z_+$. There exists a sequence $\{\tilde{g}_n\}_{n=1}^{\infty} \subset R\mathcal{H}^\infty$ of outer functions (which depend on N) with the following properties:

(1) $\lim_{n\to\infty} \|\tilde{g}_n - g_N\|_{\mathcal{H}^\infty} = 0$.
(2) For each $A \in Z_+$,

$$\lim_{n\to\infty} \|f_A(1 - \tilde{g}_n)\|_{\mathcal{H}^\infty} = 0.$$

(3) $\tilde{g}_m^{-1}\tilde{g}_n \in \mathcal{H}^\infty \cap \mathcal{C}_e$ for each $m, n \in Z_+$.

(4) The only imaginary zeros of $\tilde{g}_n$ are the points $\{\omega_k\}_{k=0}^{l}$; each with multiplicity 1.

We shall refer to $\{\tilde{g}_n\}_{n=1}^{\infty}$ as an *inverting sequence* for $T_{2_{n_o}}$, which we shall be construct below.

∎

Proof The proof follows by inspection of the function

$$g_N \stackrel{\text{def}}{=} \left(\frac{s}{s+\frac{1}{N}}\right)^{m_0} \prod_{k=1}^{l-1} \left(\frac{s^2+\omega_k^2}{s^2+2\frac{1}{N}s+\omega_k^2+\frac{1}{N}^2}\right)^{m_k} \left(\frac{N}{s+N}\right)^{m_l}$$

and the use of proposition 2.10.1. The idea is to factor out the essential information (i.e. the imaginary zeros) and approximate the rest by minimum phase $R\mathcal{H}^\infty$ functions. ∎

We now show how to construct the sequence $\{T_{2_n}\}_{n=1}^{\infty}$ so that it approximates T_2 in a manner which makes the associated finite dimensional model matching problem an appropriate tool for studying the infinite dimensional problem posed in definition 5.2.1. The construction is such that for large n, $T_{2_{n_o}}$ contains all the "essential information". More specifically, $T_{2_{n_i}}$ will approximate T_{2_i} uniformly on compact frequency intervals, $T_{2_{n_o}}$ will approximate T_{2_o} uniformly, and $\{\tilde{g}_n\}_{n=1}^{\infty}$ will be an inverting sequence for $T_{2_{n_o}}$. The construction presented shall be exploited in later chapters on $\mathcal{H}^\infty$ design.

Construction 5.5.1 (Construction of $\{T_{2_n}\}_{n=1}^{\infty}$)
Let $\{T_{2_{n_i}}\}_{n=1}^{\infty} \subset R\mathcal{H}^\infty$ be any inner sequence such that

$$\lim_{n\to\infty} \left\|(T_{2_{n_i}} - T_{2_i})X_{[-\Omega,\Omega]}\right\|_{\mathcal{H}^\infty} = 0$$

for each $\Omega \in R_+$, however large. Let $\{A_n\}_{n=1}^{\infty} \subset R\mathcal{H}^\infty$ be any minimum phase sequence such that

$$\lim_{n\to\infty} \left\|A_n - T_{2_o}g_N^{-1}\right\|_{\mathcal{H}^\infty} = 0.$$

Here, N can be any positive integer. We then define

$$T_{2_{n_o}} \stackrel{\text{def}}{=} A_n\tilde{g}_n$$

and

$$T_{2_n} \stackrel{\text{def}}{=} T_{2_{n_o}}T_{2_{n_i}}.$$

Given the above, we have that $\{T_{2_n}\}_{n=1}^{\infty}$ is a uniformly bounded sequence which satisfies the following

$$\lim_{n\to\infty} \|T_{2_{n_o}} - T_{2_o}\|_{\mathcal{H}^\infty} = 0 \qquad (1)$$

$$\sup_n \left\|T_{2_{n_o}}^{-1}\tilde{g}_B\right\|_{\mathcal{H}^\infty} \le M_B < \infty \qquad (2)$$

for each $B \in Z_+$ and

$$\lim_{n\to\infty} \left\|(T_{2_n} - T_2)X_{[-\Omega,\Omega]}\right\|_{\mathcal{H}^\infty} = 0 \qquad (3)$$

for each $\Omega \in R_+$.

■

Proof

(1) follows from the following inequality

$$\|T_{2_{n_o}} - T_{2_o}\|_{\mathcal{H}^\infty} = \|\tilde{g}_n A_n - T_{2_o}\|_{\mathcal{H}^\infty} \leq \|A_n\|_{\mathcal{H}^\infty} \|\tilde{g}_n - g_N\|_{\mathcal{H}^\infty} + \|g_N\|_{\mathcal{H}^\infty} \left\|A_n - T_{2_o} g_N^{-1}\right\|_{\mathcal{H}^\infty}$$

and proposition 5.5.1. We note that (1) implies that the sequence $\{T_{2_{n_o}}\}_{n=1}^{\infty}$ is uniformly bounded. Since $T_{2_{n_i}}$ is inner, it also follows that the sequence $\{T_2\}_{n=1}^{\infty}$ is also uniformly bounded.

By construction, $T_{2_o} g_N^{-1}$ is bounded away from zero (cf. proposition 5.2.4). Moreover, it is minimum phase. Consequently, by proposition 2.10.1, there exists a minimum phase sequence $\{A_n\}_{n=1}^{\infty}$ which uniformly approximates $T_{2_o} g_N^{-1}$. Since A_n converges uniformly to $T_{2_o} {g_N}^{-1}$, it follows that A_n will be bounded away from zero for sufficiently large n. Given this, we obtain (2) from

$$\left\|T_{2_{n_o}}^{-1} \tilde{g}_B\right\|_{\mathcal{H}^\infty} \leq \left\|\tilde{g}_N^{-1} \tilde{g}_B\right\|_{\mathcal{H}^\infty} \left\|{A_n}^{-1}\right\|_{\mathcal{H}^\infty}$$

and proposition 5.5.1.

Finally, (3) follows from (1), the following inequality

$$\left\|(T_{2_n} - T_2) X_{[-\Omega,\Omega]}\right\|_{\mathcal{H}^\infty} \leq \|T_{2_{n_o}}\|_{\mathcal{H}^\infty} \left\|(T_{2_{n_i}} - T_{2_i}) X_{[-\Omega,\Omega]}\right\|_{\mathcal{H}^\infty} + \|T_{2_i}(T_{2_{n_o}} - T_{2_o})\|_{\mathcal{H}^\infty}$$

and the fact that the sequence $\{T_{2_{n_o}}\}_{n=1}^{\infty}$ is uniformly bounded. ■

Comment 5.5.1 (Practicality)
The condition on the sequence $\{T_{2_{n_i}}\}_{n=1}^{\infty}$ is reasonable. Inner functions in $\mathcal{H}^\infty$ can "usually" be approximated on compact subsets, by functions in $R\mathcal{H}^\infty$. If T_{2_i} is a delay, for example, one can use Pade' approximations to obtain the approximants $T_{2_{n_i}}$. See example 2.10.2.

■

The following proposition, loosely speaking, shows that, for large n, μ_n is bounded from above by μ_o. It should be interpreted as an upper-semicontinuity result [53, pp. 345].

Proposition 5.5.2 (Upper-semicontinuity)

$$\lim_{n\to\infty} \mu_n \leq \mu_o.$$

■

Proof The proof of this proposition follows from the following inequality

$$\mu_n \leq \|T_{1_n} - T_{2_n} Q_o\|_{\mathcal{H}^\infty} \leq \|T_1 - T_2 Q_o\|_{\mathcal{H}^\infty} + \|T_{1_n} - T_1\|_{\mathcal{H}^\infty} + \|(T_2 - T_{2_n}) Q_o\|_{\mathcal{H}^\infty}$$

which holds for each $Q_o \in \mathcal{H}^\infty$. By definition of μ_o, there exists $Q_o \in \mathcal{H}^\infty$ such that $\|T_1 - T_2 Q_o\|_{\mathcal{H}^\infty} \leq \mu_o + \epsilon$. This takes care of the first term on the right hand side of the inequality. The second term can be made arbitraily large by taking n sufficiently large. This is because T_{1_n} uniformly approximates T_1. The third term can also be made arbitrarily small by taking n sufficiently large. This follows becase T_{2_n} is a compact approximant for T_2 and because Q_o rolls-off. See lemma 2.10.1. Consequently, there exists $N \stackrel{\text{def}}{=} N(\epsilon, T_1, T_2) \in Z_+$ such that

$$\lim_{n\to\infty} \mu_n \leq \mu_o + 3\epsilon$$

for all $n \geq N$. This, however, proves the result. ∎

Comment 5.5.2 (Uniform Boundedness)
Since $\mu_o \leq \|T_1\|_{\mathcal{H}^\infty}$, we know that μ_o is bounded. Given this, the above proposition implies that μ_n is uniformly bounded. This is also known from the fact that $\mu_n \leq \|T_{1_n}\|_{\mathcal{H}^\infty}$, and T_{1_n} uniformly approximates T_1.

∎

The following theorem shows how to construct near-optimal solutions for the finite dimensional model matching problem defined by definition 5.5.1.

Theorem 5.5.1 (A "Sub-Optimal" Real-Rational Sequence)

There exists a sequence $\{\tilde{Q}_n\}_{n=1}^{\infty} \in R\mathcal{H}_0^\infty$ which is uniformly bounded, uniformly rolls-off, and such that

$$\mu_n \leq \left\|T_{1_n} - T_{2_n}\tilde{Q}_n\right\|_{\mathcal{H}^\infty} \leq \mu_n + \epsilon$$

for n sufficiently large.

∎

Proof From proposition 5.2.1, there exists $Z_n \in R\mathcal{H}^\infty$ such that

$$\|Y_n\|_{\mathcal{H}^\infty} = \min_{Z \in R\mathcal{H}^\infty} \left\|T_{1_n} - T_{2_{n_i}} Z\right\|_{\mathcal{H}^\infty}$$

where $Y_n \stackrel{\text{def}}{=} T_{1_n} - T_{2_{n_i}} Z_n$. Moreover such a Z_n is easy to compute since T_{1_n} and T_{2_n} are real-rational [23]. Now define

$$Q_{a,b,n} \stackrel{\text{def}}{=} T_{2_{n_o}}^{-1} Z_n f_a \tilde{g}_b$$

where f_a is as in proposition 5.2.3, $\tilde{g}_b$ is as in proposition 5.5.1. We first show that $Q_{a,b,n}$ is near-optimal for appropriate choice of a and b.

Step 1: Near-Optimality of $Q_{A,B,n}$.

Consider the following inequality.

$$\|T_{1_n} - T_{2_n} Q_{a,b,n}\|_{\mathcal{H}^\infty} = \left\|T_{1_n} - T_{2_{n_i}} Z_n f_a \tilde{g}_b\right\|_{\mathcal{H}^\infty} \leq \left\|T_{1_n} - T_{2_{n_i}} Z_n f_a\right\|_{\mathcal{H}^\infty} + \left\|T_{2_{n_i}} Z_n f_a (1 - \tilde{g}_b)\right\|_{\mathcal{H}^\infty}$$

$$\leq \|T_{1_n} + (Y_n - T_{1_n}) f_a\|_{\mathcal{H}^\infty} + \left\|T_{2_{n_i}} Z_n f_a (1 - \tilde{g}_b)\right\|_{\mathcal{H}^\infty}$$

$$\leq \|T_1 + (Y_n - T_1) f_a\|_{\mathcal{H}^\infty} + \|T_{1_n} - T_1\|_{\mathcal{H}^\infty} + \|(T_1 - T_{1_n}) f_a\|_{\mathcal{H}^\infty} + \left\|T_{2_{n_i}} Z_n f_a (1 - \tilde{g}_b)\right\|_{\mathcal{H}^\infty}.$$

From theorem 5.2.1, we know that there exists $A \stackrel{\text{def}}{=} A(\epsilon, T_1) \in Z_+$ (independent of n)such that

$$\|T_1 + (Y_n - T_1) f_A\|_{\mathcal{H}^\infty} \leq \max\{|T_1(j\infty)|, \max_k |T_1(j\omega_k)|, \|Y_n\|_{\mathcal{H}^\infty}\} + \epsilon$$

for all $n \in Z_+$.

By construction

$$\mu_n = \max\{|T_{1_n}(j\infty)|, \max_k |T_{1_n}(j\omega_k)|, \|Y_n\|_{\mathcal{H}^\infty}\}.$$

Since T_{1_n} uniformly approximates T_1, there exists $N_1 \stackrel{\text{def}}{=} N_1(\epsilon, T_1) \in Z_+$ such that

$$\max\{|T_1(j\infty)|, \max_k |T_1(j\omega_k)|, \|Y_n\|_{\mathcal{H}^\infty}\} \leq \mu_n + \epsilon$$

for all $n \geq N_1$. Given this, the previous inequality becomes

$$\|T_1 + (Y_n - T_1)f_A\|_{\mathcal{H}^\infty} \leq \mu_n + \epsilon$$

for all $n \geq N_1$. From the first inequality, this then gives

$$\|T_{1_n} - T_{2_n}Q_{A,b,n}\|_{\mathcal{H}^\infty} \leq \mu_n + \epsilon + \|T_{1_n} - T_1\|_{\mathcal{H}^\infty} + \|(T_1 - T_{1_n})f_A\|_{\mathcal{H}^\infty} + \left\|T_{2_{n_i}} Z_n f_A(1 - \tilde{g}_b)\right\|_{\mathcal{H}^\infty}.$$

The second and third norms on the right hand side can be made arbitrarily small by taking n sufficiently large. This follows because T_{1_n} uniformly approximates T_1 and because f_A is a fixed $\mathcal{H}^\infty$ function.

To make the last term arbitrarily small, we begin by noting that $T_{2_{n_i}} Z_n$ is uniformly bounded in $\mathcal{H}^\infty$. This follows from the inequality

$$\left\|T_{2_{n_i}} Z_n\right\|_{\mathcal{H}^\infty} \leq \min_{Z \in R\mathcal{H}^\infty} \left\|T_{1_n} - T_{2_{n_i}} Z\right\|_{\mathcal{H}^\infty} + \|T_{1_n}\|_{\mathcal{H}^\infty} \leq 2\|T_{1_n}\|_{\mathcal{H}^\infty},$$

and the fact that T_{1_n} is uniformly bounded. Given this, proposition 5.5.1 implies that b can be chosen sufficiently large to make the last term arbitrarily small. Cosequently, there exists $B \stackrel{\text{def}}{=} B(\epsilon, A) \in Z_+$ (independent of n) such that

$$\left\|T_{2_{n_i}} Z_n f_A(1 - \tilde{g}_B)\right\|_{\mathcal{H}^\infty} \leq \epsilon$$

for all $n \in Z_+$.

Combining the above, we have that there exists $N \geq N_1$ in Z_+ such that

$$\|T_{1_n} - T_{2_n}Q_{A,B,n}\|_{\mathcal{H}^\infty} \leq \mu_n + 4\epsilon$$

for all $n \geq N$. This shows that $Q_{A,B,n}$ is near-optimal for large sufficiently large n.

Step 2: Uniform Boundedness.

We now show that $\{Q_{A,B,n}\}_{n=1}^\infty$ is uniformly bounded in $\mathcal{H}^\infty$. We obsernve that $Z_n f_A$ is uniformly bounded since Z_n is and since f_A is a fixed $\mathcal{H}^\infty$ function. We then note that

$$\sup_n \left\|T_{2_{n_o}}^{-1} \tilde{g}_B\right\|_{\mathcal{H}^\infty} \leq M_B < \infty$$

for each $B \in Z_+$, from construction 5.5.1. These imply the uniform boundedness of $\{Q_{A,B,n}\}_{n=1}^\infty$.

Step 3: Uniform Roll-off.

That $\{Q_{A,B,n}\}_{n=1}^\infty$ is also uniformly rolls-off follows since $T_{2_{n_o}}^{-1} Z_n \tilde{g}_B$ is uniformly bounded and since f_A rolls-off (cf. proposition 5.2.3).

Step 4: Construction of $\tilde{Q}_n$.

Finally, we note that f_A is the only irrational function used in constructing $Q_{A,B,n}$. However, by proposition 5.2.3, $f_A \in \mathcal{C}_e$. Hence, from proposition 2.10.1, f_A can be uniformly approximated by $R\mathcal{H}^\infty$ functions which possess similar properties. The construction of the sequence $\{\tilde{Q}_n\}_{n=1}^{\infty}$ thus follows. ∎

The following theorem guarantees the converse of proposition 5.5.2. It shows that μ_n approaches μ_o as the approximants get better.

Theorem 5.5.2 ("Continuity" Result)

$$\lim_{n\to\infty} \mu_n = \mu_o.$$

∎

Proof Proposition 5.5.2 gives upper-semicontinuity:

$$\lim_{n\to\infty} \mu_n \le \mu_o.$$

We now prove lower-semicontinuity. We begin with the following inequality

$$\mu_o \le \left\|T_1 - T_2\tilde{Q}_n\right\|_{\mathcal{H}^\infty} \le \left\|T_{1_n} - T_{2_n}\tilde{Q}_n\right\|_{\mathcal{H}^\infty} + \|T_1 - T_{1_n}\|_{\mathcal{H}^\infty} + \left\|(T_{2_n} - T_2)\tilde{Q}_n\right\|_{\mathcal{H}^\infty}$$

where $\tilde{Q}_n$ is constructed as in theorem 5.5.1.

The first term can be made arbitrarily close to μ_n by constructing $\tilde{Q}_n$ appropriately.

The second term can be made arbitrarily small by taking n sufficently large. This is because T_{1_n} uniformly approximates T_1.

Finally, the third term can be made arbitrarily large by taking n sufficiently large. This is because T_{2_n} is a compact approximant for T_2 (cf. construction 5.5.1) and because $\tilde{Q}_n$ is uniformly bounded and uniformly rolls-off (cf. lemma 2.10.1).

Consequently,

$$\mu_o \le \lim_{n\to\infty} \mu_n$$

for n sufficiently large. This proves the converse inequality (lower-semicontinuity), and hence the theorem. ∎

Comment 5.5.3 (The case $T_{1_n} = T_1$)
Upon inspection of the above proof, we see that if $T_{1_n} = T_1$, then the above result would hold for all $n \in Z_+$. Otherwise, the result is only guaranteed for sufficiently large n. This is due to the way in which we construct $\tilde{Q}_n$. It should be apparent that a different construction could be guaranteed to be near-optimal for all n. However, it is not obvious that such a construction would be uniformly bounded and would uniformly roll-off.

∎

From the above theorems we get the following corollary.

Corollary 5.5.1 (A Near-Optimal Real-Rational Solution)

There exists a seqence $\{\tilde{Q}_n\}_{n=1}^{\infty} \in R\mathcal{H}_0^{\infty}$ *which is uniformly bounded, uniformly rolls-off, and such that*

$$\mu_o \leq \left\| T_1 - T_2\tilde{Q}_n \right\|_{\mathcal{H}^\infty} \leq \mu_o + \epsilon$$

for each n *sufficiently large.*

∎

Proof Here we take $\{\tilde{Q}_n\}_{n=1}^{\infty}$ as constructed in theorem 5.5.1. The lower inequality is obvious since $\tilde{Q}_n \in R\mathcal{H}_0^{\infty}$ and is thus admissible.

The prove the upper inequality, we consider the following inequality

$$\left\| T_1 - T_2\tilde{Q}_n \right\|_{\mathcal{H}^\infty} \leq \left\| T_{1_n} - T_{2_n}\tilde{Q}_n \right\|_{\mathcal{H}^\infty} + \| T_1 - T_{1_n} \|_{\mathcal{H}^\infty} + \left\| (T_2 - T_{2_n})\tilde{Q}_n \right\|_{\mathcal{H}^\infty}.$$

We then note that that the last two terms can be made arbitrarily small by taking n sufficiently large. The result then follows from the fact that

$$\left\| T_{1_n} - T_{2_n}\tilde{Q}_n \right\|_{\mathcal{H}^\infty} \leq \mu_n + \epsilon,$$

for n sufficiently large, and theorem 5.5.2. ∎

5.6 Summary

In this chapter we defined the $\mathcal{H}^\infty$ *Model Matching Problem* and showed how to construct near-optimal solutions for it. Sequences of appropriately formulated finite dimensional $\mathcal{H}^\infty$ *Model Matching Problems* were also considered in an effort to avoid the complex infinite dimensional $\mathcal{H}^\infty$ *Model Matching Problem.* It was shown that near-optimal real-rational solutions could be constructed for the infinite dimensional $\mathcal{H}^\infty$ *Model Matching Problem* from those resulting from the finite dimensional problems. The ideas presented in the construction shall be heavily exploited in subsequent chapters on $\mathcal{H}^\infty$ design.

Chapter 6

Design via $\mathcal{H}^\infty$ Sensitivity Optimization

6.1 Introduction

In this chapter we consider the problem of designing near-optimal finite dimensional compensators for infinite dimensional plants via $\mathcal{H}^\infty$ sensitivity optimization. Such an approach can be motivated by design specifications which require some specified degree of robustness or $\mathcal{L}^2$ disturbance rejection. A systematic procedure is presented. More specifcally, we provide a solution to the $\mathcal{H}^\infty$ *Approximate/Design Sensitivity Problem*, the $\mathcal{H}^\infty$ *Purely Finite Dimensional Sensitivity Problem*, and the $\mathcal{H}^\infty$ *Loop Convergence Sensitivity Problem.*

6.2 $\mathcal{H}^\infty$ Approximate/Design Sensitivity Problem

In this section we present some definitions and assumptions to precisely state the $\mathcal{H}^\infty$ *Approximate/Design Sensitivity Problem.* Notation to be used throughout the chapter is also established.

Throughout the chapter we shall be working over the ring $\mathcal{H}^\infty$. A transfer function will be called stable if and only if it belongs to $\mathcal{H}^\infty$. Given this, $\mathcal{F}(\mathcal{H}^\infty)$ will denote the fraction field associated with $\mathcal{H}^\infty$. $\mathcal{F}_c(\mathcal{H}^\infty)$ will denote those elements of $\mathcal{F}(\mathcal{H}^\infty)$ which can be represented as the ratio of coprime factors in $\mathcal{H}^\infty$.

It has been shown in [52] that $\mathcal{F}_c(\mathcal{H}^\infty)$ consists of all elements in $\mathcal{F}(\mathcal{H}^\infty)$ which are stabilizable. With this in mind, we make the following assumption about the infinite dimensional plant P.

Assumption 6.2.1 (Permissible Plants)

$P \in \mathcal{F}_c(\mathcal{H}^\infty)$ [1].

∎

This assumption permits us to exploit the algebraic ideas presented in chapter 3. More specifically, the assumption allows us to associate a coprime factorization $(N_p, D_p) \in \mathcal{H}^\infty$ for P, over $\mathcal{H}^\infty$, with Bezout factors $(N_k, D_k) \in \mathcal{H}^\infty$. That is,

$$P = \frac{N_p}{D_p}$$

[1] Although not explicitly stated, the causality of P is assumed.

and

$$D_p D_k - N_p N_k = 1.$$

From proposition 3.2.1, it then follows that the set of all compensators which internally stabilize P, with respect to the ring $\mathcal{H}^\infty$, are parameterized by

$$K(P,Q) \stackrel{\text{def}}{=} \frac{N_k - D_p Q}{D_k - N_p Q}$$

where Q is any element in $\mathcal{H}^\infty$. The following assumption is made regarding N_k and D_k.

Assumption 6.2.2 (Nominal Compensator)

(1) $N_k \in \mathcal{H}_0^\infty \cap \mathcal{C}_e$.
(2) $D_k \in \mathcal{C}_e$.
(3) Both N_k and D_k are known.

∎

Comment 6.2.1 (Practicality, Computational Issues)
Since we ultimately want a near-optimal strictly proper compensator, it makes sense to let the *nominal compensator* $K_{nom} \stackrel{\text{def}}{=} \frac{N_k}{D_k}$ be strictly proper. Having $N_k \in \mathcal{H}_0^\infty$ is thus justified. The continuity of N_k and D_k is needed so that they mey be approximated uniformly by $R\mathcal{H}^\infty$ functions (cf. proposition 2.10.1). Assumption 6.2.2 is thus made with little loss of generality. The fact that we need to know the pair N_k and D_k is not very demanding. Since we seek a near-optimal strictly proper compensator, we should be able to come up with one strictly proper stabilizing compensator.

We do acknowlegde that, in general, the computation of N_p, D_p, N_k, and D_k may be difficult. This issue shall not be addressed in this thesis.

∎

Given that $N_k \in \mathcal{H}_0^\infty$, it follows that if we allow Q to vary over $\mathcal{H}_0^\infty$, then we get all strictly proper compensators which internally stabilize P. We shall be doing this throughout the section; i.e. all infimizations involving P shall be carried out over $\mathcal{H}_0^\infty$.

In this chapter we shall construct a sequence of real-rational approximants P_n for P. They will be constructed such that they are elements of $\mathcal{F}_c(\mathcal{H}^\infty)$. Given this, we are guaranteed the existence of a coprime factorization $(N_{p_n}, D_{p_n}) \in R\mathcal{H}^\infty$ for P_n, over $R\mathcal{H}^\infty$, with Bezout factors $(N_{k_n}, D_{k_n}) \in R\mathcal{H}^\infty$ [56]. That is,

$$P_n = \frac{N_{p_n}}{D_{p_n}}$$

and

$$(D_{p_n} D_{k_n} - N_{p_n} N_{k_n})^{-1} \in R\mathcal{H}^\infty.$$

The functions P_n, N_{p_n}, D_{p_n}, N_{k_n}, and D_{k_n} shall be constructed below. From proposition 3.2.1, it then follows that the set of all compensators which internally stabilize P_n, with respect to the ring $\mathcal{H}^\infty$, are parameterized by

$$K(P_n,Q) \stackrel{\text{def}}{=} \frac{N_{k_n} - D_{p_n} Q}{D_{k_n} - N_{p_n} Q}$$

where Q is any element in $\mathcal{H}^\infty$.

The function N_{k_n} will be constructed so that it rolls-off. Given this, it follows that if we allow Q to vary over $\mathcal{H}_0^\infty$, then we get all strictly proper compensators which internally stabilize P_n. We shall be doing this throughout the section; i.e. all infimizations which involve P_n shall be carried out over $\mathcal{H}_0^\infty$.

Throughout the chapter, we shall assume the following inner-outer factorizations over $\mathcal{H}^\infty$:

$$N_p = N_{p_i} N_{p_o} \qquad D_p = D_{p_i} D_{p_o}$$

$$N_{p_n} = N_{p_{n_i}} N_{p_{n_o}} \qquad D_{p_n} = D_{p_{n_i}} D_{p_{n_o}}.$$

Such factorizations are guaranteed to exist by proposition 2.7.6.

Comment 6.2.2 (Computational Issues)
It should be noted that the computation of inner-outer factorizations for N_p and D_p may be difficult in practice. Such factorizations, however, can be obtained for a large class of plants. The assumption that inner-outer factorizations for N_p and D_p are known is thus justified. This issue shall not receive further consideration in this thesis.

∎

In this section we shall formulate an $\mathcal{H}^\infty$ weighted sensitivity problem. To do so, we shall require a frequency dependent *weighting function* W. The following assumption shall be made on W.

Assumption 6.2.3 (Weighting Function)

(1) $W \in R\mathcal{H}^\infty$.
(2) W is outer.

∎

Here, W may be proper or strictly proper.

We now define the notion of an $\mathcal{H}^\infty$*-sensitivity measure* as follows.

Definition 6.2.1 ($\mathcal{H}^\infty$-Sensitivity Measure)
Let $Q \in \mathcal{H}^\infty$ and $F, G, \in \mathcal{F}_c(\mathcal{H}^\infty)$. Also, let $K(G,Q)$ denote a compensator which internally stabilizes G with respect to the ring $\mathcal{H}^\infty$. If it also internally stabilizes F, it is appropriate to define the $\mathcal{H}^\infty$-sensitivity measure of the pair (F, $K(G,Q)$) as follows:

$$J_{\mathcal{H}^\infty}(F, K(G,Q)) \stackrel{\text{def}}{=} \left\| \frac{W}{1 - FK(G,Q)} \right\|_{\mathcal{H}^\infty}.$$

∎

From definition 8.2.1 above, it follows that $J_{\mathcal{H}^\infty}(P, K(P,Q)) \stackrel{\text{def}}{=} \left\| \frac{W}{1-PK(P,Q)} \right\|_{\mathcal{H}^\infty}$. Substituting into definition 4.3.1 and allowing Q to vary over $\mathcal{H}_0^\infty$, then gives us the following expression for the *optimal performance*, μ_{opt}.

Definition 6.2.2 (Optimal Performance)

$$\mu_{opt} \stackrel{\text{def}}{=} \inf_{Q \in \mathcal{H}_0^\infty} \left\| \frac{W}{1 - PK(P,Q)} \right\|_{\mathcal{H}^\infty} = \inf_{Q \in \mathcal{H}_0^\infty} \|T_1 - T_2 Q\|_{\mathcal{H}^\infty}$$

where $T_1 \stackrel{\text{def}}{=} W D_p D_k$ and $T_2 \stackrel{\text{def}}{=} W D_p N_p$.

∎

We emphasize that this definition defines an infinite dimensional optimization problem; one which we want to, and will, avoid solving. Moreover, we note that the problem is an $\mathcal{H}^\infty$ *Model Matching Problem*, such as the one studied in section 5.2.

Similarly, from definition 8.2.1, it follows that $J_{\mathcal{H}^\infty}(P_n, K(P_n, Q)) \stackrel{\text{def}}{=} \left\| \frac{W}{1-P_nK(P_n,Q)} \right\|_{\mathcal{H}^\infty}$. After substituting this into definition 5.5.1, and allowing Q to vary over $R\mathcal{H}_0^\infty$, one obtains the following expression for the *expected performance*, μ_n.

Definition 6.2.3 (Expected Performance)

$$\mu_n \stackrel{\text{def}}{=} \inf_{Q \in R\mathcal{H}_0^\infty} \left\| \frac{W}{1 - P_nK(P_n, Q)} \right\|_{\mathcal{H}^\infty} = \inf_{Q \in R\mathcal{H}_0^\infty} \|T_{1_n} - T_{2_n}Q\|_{\mathcal{H}^\infty}$$

where $T_{1_n} \stackrel{\text{def}}{=} WD_{p_n}D_{k_n}$ *and* $T_{2_n} \stackrel{\text{def}}{=} WD_{p_n}N_{p_n}$.

∎

Here we can infimize over $R\mathcal{H}_0^\infty$ since W and P_n are real-rational [23]. We note that this definition defines a sequence of finite dimensional model matching problems.

An optimal or near-optimal solution to this problem is typically found by first considering the "inner problem" [23]:

$$\inf_{Z \in R\mathcal{H}^\infty} \left\| T_{1_n} - T_{2_{n_i}}Z \right\|_{\mathcal{H}^\infty}.$$

Here, $T_{2_{n_i}} = D_{p_{n_i}}N_{p_{n_i}}$ is the inner part of $T_{2_n} \stackrel{\text{def}}{=} WD_{p_n}N_{p_n}$ (recall that W is outer). From proposition 5.2.1, we know that there exists $Z_n \in R\mathcal{H}^\infty$ such that

$$\left\| T_{1_n} - T_{2_{n_i}}Z_n \right\|_{\mathcal{H}^\infty} = \min_{Z \in R\mathcal{H}^\infty} \left\| T_{1_n} - T_{2_{n_i}}Z \right\|_{\mathcal{H}^\infty}.$$

Moreover, Z_n is unique [23]. Lets define

$$Q_n \stackrel{\text{def}}{=} T_{2_{n_o}}^{-1} Z_n$$

where $T_{2_{n_o}} = WD_{p_{no}}N_{p_{no}}$ is the outer part of T_{2_n}. This Q_n generates a finite dimensional compensator

$$K_n \stackrel{\text{def}}{=} K(P_n, Q_n) = \frac{N_{k_n} - D_{p_n}Q_n}{D_{k_n} - N_{p_n}Q_n}.$$

This compensator, as we shall see (section 6.3), need not even stabilize P. We thus need to modify it. For this reason, we define a *roll-off operator*

$$r : Q_n \to \tilde{Q}_n \in R\mathcal{H}_0^\infty.$$

To construct r we shall exploit theorem 5.5.1. The exact form of r will be determined subsequently.

The compensator generated by $\tilde{Q}_n$ is given by

$$\tilde{K}_n \stackrel{\text{def}}{=} K(P_n, \tilde{Q}_n) = \frac{N_{k_n} - D_{p_n}\tilde{Q}_n}{D_{k_n} - N_{p_n}\tilde{Q}_n}.$$

Given this, we consider the feedback system obtained by substituting $\tilde{K}_n$ into a closed loop system with the infinite dimensional plant P. Let $H(P, \tilde{K}_n)$ denote the resulting closed loop transfer function matrix from r, d to e, u. We then have

$$\begin{bmatrix} e \\ u \end{bmatrix} = H(P, \tilde{K}_n) \begin{bmatrix} r \\ d \end{bmatrix},$$

where

$$H(P, \tilde{K}_n) = \begin{bmatrix} \frac{1}{1-P\tilde{K}_n} & \frac{P}{1-P\tilde{K}_n} \\ \frac{\tilde{K}_n}{1-P\tilde{K}_n} & \frac{1}{1-P\tilde{K}_n} \end{bmatrix}.$$

Substituting for $\tilde{K}_n$, then gives

$$H(P, \tilde{K}_n(\tilde{Q}_n)) = \begin{bmatrix} D_p(D_{k_n} - N_{p_n}\tilde{Q}_n) & N_p(D_{k_n} - N_{p_n}\tilde{Q}_n) \\ D_p(N_{k_n} - D_{p_n}\tilde{Q}_n) & D_p(D_{k_n} - N_{p_n}\tilde{Q}_n) \end{bmatrix} \frac{1}{\delta(P, \tilde{K}_n(\tilde{Q}_n))}$$

where

$$\delta(P, \tilde{K}_n(\tilde{Q}_n)) \stackrel{\text{def}}{=} D_p(D_{k_n} - N_{p_n}\tilde{Q}_n) - N_p(N_{k_n} - D_{p_n}\tilde{Q}_n).$$

Given that internal stability can be shown, the *actual performance*, $\tilde{\mu}_n$ defined in definition 4.3.3 is well defined and becomes:

Definition 6.2.4 (Actual Performance)

$$\tilde{\mu}_n \stackrel{\text{def}}{=} \left\| \frac{W}{1 - P\tilde{K}_n} \right\|_{\mathcal{H}^\infty} = \left\| \frac{W(D_p - D_{p_n})(D_{k_n} - N_{p_n}\tilde{Q}_n) + (T_{1_n} - T_{2_n}\tilde{Q}_n)}{\delta(P, \tilde{K}_n(\tilde{Q}_n)} \right\|_{\mathcal{H}^\infty}$$

∎

Given the above definitions, the $\mathcal{H}^\infty$ *Approximate/Design Sensitivity Problem* then becomes to find conditions on the approximants $\{P_n\}_{n=1}^{\infty}$, and on the *roll-off operator* r, such that the *actual performance* approaches the *optimal performance*; i.e.

$$\lim_{n \to \infty} \tilde{\mu}_n = \mu_{opt}.$$

Equivalently, this problem can be viewed as that of finding a near-optimal compensator for the infinite dimensional plant P. The problem also addresses the question: What is a "good" finite dimensional approximant?

Because this problem is of primary concern in this research, we now indicate what difficulties are associated with the problem.

6.3 Why is the Approximate/Design Problem Hard?

There are several reasons one can give to illustrate the difficulties associated with the *Approximate/Design Problem.* We now discuss some of these.

First, one must note that the weighted $\mathcal{H}^\infty$ sensitivity problem, in general, is discontinuous with respect to plant perturbations, even when the uniform topology on $\mathcal{H}^\infty$ is imposed. This has been demonstrated in [53]. Consequently, simple continuity arguements cannot be used. It must also be noted, however, that even if it were continuous in the uniform topology, there are many infinite dimensional plants which cannot be approximated uniformly by real-rational functions (e.g. a delay; see proposition 2.10.1).

A second difficulty can be attributed to the fact that the associated Hankel operator is often not compact. When this is the case it cannot be approximated uniformly by finite rank operators, and again it is not clear how to proceed. How to approximate a non-compact operator then becomes the non-trivial issue.

Another difficulty can be attributed to the fact that weighted $\mathcal{H}^\infty$ optimal solutions generally exhibit bad properties. More specifically, one can show that the optimal solution is often unbounded

and results in an improper compensator. One can correctly argue that this is usually an existence issue, nevertheless, it is an issue which a designer must contend with.

The following example illustrates that even uniform approximations can lead to bad results.

Example 6.3.1 (Discontinuity, Instability, "Open-loop Intuition")
Let our infinite dimensional plant be given by $P(s) = \frac{e^{-s}}{s+1}$. Let the weighting function be given by $W = \frac{s+1}{s+\beta}$, where $0 < \beta < 1$. The associated optimal compensator is infinite dimensional. It can be found by solving the infinite dimensional model matching problem defined by:

$$\mu_{opt} \stackrel{\text{def}}{=} \inf_{Q \in \mathcal{H}_0^\infty} \|W(1 - PQ)\|_{\mathcal{H}^\infty} .$$

This has been done in [14]. There, it is shown that

$$\mu_{opt} = \max\{\ |W(j\infty)|,\ \|\Gamma_{We^s}\|\ \} = \|\Gamma_{We^s}\|.$$

Thus, $\|\Gamma_{We^s}\| > 1$.

We want to obtain a near-optimal finite dimensional compensator, by solving an appropriately formulated finite dimensional problem. Let $P_n(s) = (\frac{n}{s+n})^n \frac{1}{s+1}$ define a set of finite dimensional approximants for P. It can be shown that P_n uniformly aprroximates P on the extended imaginary axis (cf. example 2.10.1). The approximants P_n are thus terrific, based on "open-loop intuition".

A solution to the finite dimensional optimization problem

$$\mu_n \stackrel{\text{def}}{=} \inf_{Q \in R\mathcal{H}_0^\infty} \|W(1 - P_nQ)\|_{\mathcal{H}^\infty}$$

is given by

$$Q_n = (1-\beta)\,(1 + \frac{s}{n})^n.$$

This Q_n results in $\mu_n = |W(j\infty)| = 1$. We note that μ_n cannot approach μ_{opt} for large n since $\mu_{opt} > 1 = \mu_n$ for all $n \in Z_+$. This has been also noted in [53]. The example thus illustrates the discontinuity of weighted $\mathcal{H}^\infty$ sensitivity measures.

The above Q_n generates a finite dimensional compensator

$$K_n \stackrel{\text{def}}{=} \frac{-Q_n}{1 - P_nQ_n} = (\beta - 1)\, W(s)\, (1 + \frac{s}{n})^n.$$

Given this, we note that

$$\tilde{\mu}_n \stackrel{\text{def}}{=} \left\| \frac{W}{1 - PK_n} \right\|_{\mathcal{H}^\infty} = \left\| \frac{W(1 - P_nQ_n)}{1 - (P_n - P)Q_n} \right\|_{\mathcal{H}^\infty}.$$

We also note that

$$\|(P_n - P)Q_n\|_{\mathcal{H}^\infty} > 1$$

for all $n \in Z_+$. Moreover, it can be shown that K_n does not internally stabilize the plant P for any value of $n \in Z_+$.

One might argue that this is because Q_n is improper to begin with. One might go a step further to correct the above instability problem by rolling-off Q_n with some roll-off function. This can not be done using a fixed order roll-off function. Even if we were able to guarantee stability, there is no natural way to modify Q_n so that $\tilde{\mu}_n$ is close to μ_{opt} for large n.

The difficulty lies in the approximants P_n. They are bad approximants of P, given that we are solving a weighted $\mathcal{H}^\infty$ sensitivity problem. They fail to approximate the inner part of the plant. This is crucial given that we are solving a sensitivity problem.

■

The above example clearly shows that the approximants P_n must be chosen appropriately. Approximants must be chosen on the basis of a closed loop design objective; not on open loop intuition. Consequently, which approximants are used is critically dependent on which design criterion is used.

The example also shows that the way in which we modify the resulting Q_n must be done cleverly. "Classical" fixed order roll-off functions, in general, will not work.

Finally, the example shows that simple continuity arguments can not be used to obtain a solution to our Approximate/Design problem.

We now present our solution to the $\mathcal{H}^\infty$ *Approximate/Design Sensitivity Problem.*

6.4 Solution to $\mathcal{H}^\infty$ Approximate/Design Sensitivity Problem

In this section we shall solve the $\mathcal{H}^\infty$ *Approximate/Design Sensitivity Problem.* We shall do so by constructing a near-optimal finite dimensional compensator for the infinite dimensional plant P. This will be done by appropriately modifying finite dimensional solutions $\{Q_n\}_{n=1}^\infty$ based on the finite dimensional approximants $\{P_n\}_{n=1}^\infty$. The techniques developed in Chapter 5 shall be heavily exploited.

In this section, the following assumption will be made about the infinite dimensional plant P and the finite dimensional approximants $\{P_n\}_{n=1}^\infty$.

Assumption 6.4.1 (Construction of Approximants and Bezout Factors)

(1) N_{p_o} and D_{p_o} have a finite number of zeros on the extended imaginary axis; each with finite algebraic multiplicity.

(2) The sequence $\{N_{p_{n_i}}\}_{n=1}^\infty \subset R\mathcal{H}^\infty$ consists of inner functions which uniformly approximate N_{p_i} on all compact frequency intervals (excluding the point $j\infty$); i.e. for each $\Omega \in R_+$, however large, we have $\lim_{n\to\infty} \left\| (N_{p_{n_i}} - N_{p_i}) X_{[-\Omega,\Omega]} \right\|_{\mathcal{H}^\infty} = 0$.

(3) The sequence $\{N_{p_{n_o}}\}_{n=1}^\infty \subset R\mathcal{H}^\infty$ consists of outer functions which uniformly approximate N_{p_o}; i.e. $\lim_{n\to\infty} \|N_{p_{n_o}} - N_{p_o}\|_{\mathcal{H}^\infty} = 0$. Moreover, the sequence is constructed as indicated in construction 5.5.1.

(4) The sequence $\{D_{p_{n_i}}\}_{n=1}^\infty \subset R\mathcal{H}^\infty$ consists of inner functions which uniformly approximate D_{p_i}; i.e. $\lim_{n\to\infty} \left\| D_{p_{n_i}} - D_{p_i} \right\|_{\mathcal{H}^\infty} = 0$.

(5) The sequence $\{D_{p_{n_o}}\}_{n=1}^\infty \subset R\mathcal{H}^\infty$ consists of outer functions which uniformly approximate D_{p_o}; i.e. $\lim_{n\to\infty} \|D_{p_{n_o}} - D_{p_o}\|_{\mathcal{H}^\infty} = 0$. Moreover, the sequence is constructed as indicated in construction 5.5.1.

(6) $P_n \stackrel{\text{def}}{=} \frac{N_{p_n}}{D_{p_n}}$ where $N_{p_n} \stackrel{\text{def}}{=} N_{p_{n_i}} N_{p_{n_o}}$ and $D_{p_n} \stackrel{\text{def}}{=} D_{p_{n_i}} D_{p_{n_o}}$.

(7) The sequence $\{N_{k_n}\}_{n=1}^\infty \subset R\mathcal{H}_0^\infty$ uniformly approximates N_k; i.e. $\lim_{n\to\infty} \|N_{k_n} - N_k\|_{\mathcal{H}^\infty} = 0$.

(8) The sequence $\{D_{k_n}\}_{n=1}^{\infty} \subset R\mathcal{H}^{\infty}$ uniformly approximates D_k; i.e. $\lim_{n\to\infty} \|D_{k_n} - D_k\|_{\mathcal{H}^\infty} = 0$.

∎

Comment 6.4.1 (Applicability, Practicality)

(a) Poles and Zeros on Extended Imaginary Axis.

Condition (1) simply says that P has a finite number of poles and zeros on the extended imaginary axis. Relaxing this condition will be an area for future research.

(b) The Inner Part and Right Half Plane Zeros.

Condition (2) is reasonable since it allows N_{p_i} to be discontinuous at ∞. It thus allows for plants with delays. Delays can be approximated uniformly on compact frequency intervals using Pade' approximants (cf. example 2.10.2). Such approximants agree with control engineering intuition: the need to approximate the plant at "low" frequencies.

The condition allows P to have an infinite number of zeros in the open right half plane. P , for example, may contain an infinite Blaschke product of open right half plane zeros. In such a case, the partial products can be used as the $N_{p_{n_i}}$ [30].

If P has an infinite number of open right half plane zeros, then the zeros can only accumulate at ∞. If they were to accumulate within the finite open right half plane, then this would imply that N_{p_i} is identically zero in the open right half plane (cf. proposition 2.3.5). If they were to accumulate on the imaginary axis then N_{p_i} would possess essential singularites at those points and hence we cold not approximate it uniformly on compact frequency intervals (cf. proposition 2.7.3). It thus follows that the only point of accumulation can be ∞.

(c) Continuity of N_{p_o}, D_{p_i}, and D_{p_o}.

It should be noted that the approximants in (3), (4), and (5) are guaranteed to exist if and only if $N_{p_o}, D_{p_i}, D_{p_o} \in \mathcal{C}_e$. This follows from proposition 2.10.1.

(d) Approximation of Inner and Outer Parts.

We approximate the inner and outer parts separately, in order to control the pole-zero structure of the approximants P_n on the imaginary axis. If we did not perform the approximations in the above manner, then the pole-zero structure of P_n and P may differ drastically on the imaginary axis, even for large n. Such a situation is highly undesirable since $\mathcal{H}^\infty$ sensitivity solutions are discontinuous with respect to addition of poles and zeros on the imaginary axis. Since the approximants are based on the construction given in construction 5.5.1, the ideas in construction 5.5.1 are critical.

(e) Right Half Plane Poles.

Condition (4) implicitly forces P to have only a finite number of open right half plane poles. This follows from proposition 2.7.3.

This makes sense since a plant with an infinite number of open right half plane poles cannot be stabilized by a finite dimensional compensator.

(f) Continuity of N_k and D_k.

We note that the existence of the approximants in (7) and (8) are guaranteed by assumption 6.2.2 modulo proposition 2.10.1.

∎

Comment 6.4.2 (Stable Plants)
For stable plants we may choose $N_p = P$, $N_{p_n} = P_n$, $D_p = D_{p_n} = D_k = D_{k_n} = 1$, and $N_k = N_{k_n} = 0$. In such a case we choose P_{n_i} to approximate P_i on compact frequency intervals. We also choose P_{n_o} such that it approximates P_o uniformly on the extended imaginary axis. Moreover, P_{n_o} is choosen, according to construction 5.5.1, so that its zero structure on the imaginary axis does not differ drastically from that of P_o.

∎

The following lemma shows that if P_n is constructed as above, then N_{p_n} and D_{p_n} will be a coprime factorization for P_n over $R\mathcal{H}^\infty$, with Bezout factors N_{k_n} and D_{k_n}.

Lemma 6.4.1 (Algebraic Properties of Approximants)
There exists $N \in Z_+$ such that

$$(D_{p_n} D_{k_n} - N_{p_n} N_{k_n})^{-1} \in R\mathcal{H}^\infty$$

for all $n \geq N$.

∎

Proof Adding and subtracting appropriately, gives the following equality:

$$D_{p_n} D_{k_n} - N_{p_n} N_{k_n} = (D_{p_n} - D_p)(D_{k_n} - D_k) + D_k(D_{p_n} - D_p) + D_p(D_{k_n} - D_k) + D_p D_k$$
$$-(N_{p_n} - N_p)(N_{k_n} - N_k) - N_k(N_{p_n} - N_p) - N_p(N_{k_n} - N_k) - N_p N_k.$$

However, by assumption $D_p D_k - N_p N_k = 1$. The result then follows from assumption 8.4.1. ∎

Comment 6.4.3 (Strict Propriety of N_k)
In the above proof we used the strict propriety of N_k along with the fact that N_{p_n} approximates N_p uniformly on compact frequency intervals. If N_k were only proper then we would need N_{p_n} to be a uniform approximant, say, in order to guarantee that $||N_k(N_{p_n} - N_p)||_{\mathcal{H}^\infty}$ is small for large n.

∎

With this lemma and proposition 3.2.1, it follows that the set of all compensators which internally stabilize P_n, with respect to the ring $R\mathcal{H}^\infty$, are parameterized by

$$K(P_n, Q) \stackrel{\text{def}}{=} \frac{N_{k_n} - D_{p_n} Q}{D_{k_n} - N_{p_n} Q}$$

where Q is any element in $R\mathcal{H}^\infty$ [56]. Consequently, the approximants P_n, as constructed above, possess the desired algebraic properties.

We now relate the structure of our problem to the finite dimensional $\mathcal{H}^\infty$ *Model Matching Problem* considered in section 5.5.

Proposition 6.4.1 (Structure)

$$T_{1_n} \stackrel{\text{def}}{=} W D_{p_n} D_{k_n}$$

and

$$T_{2_n} \stackrel{\text{def}}{=} W D_{p_n} N_{p_n}$$

satisfy the conditions in section 5.5.

■

Proof

Given the previous lemma, the result follows from assumption 8.4.1. We note, for example, that

$$\lim_{n\to\infty} \|T_{1_n} - T_1\|_{\mathcal{H}^\infty} = 0$$

and

$$\lim_{n\to\infty} \left\|(T_{2_n} - T_2)X_{[-\Omega,\Omega]}\right\|_{\mathcal{H}^\infty} = 0$$

for each $\Omega \in R_+$, however large. These follow from the inequalities

$$\|T_{1_n} - T_1\|_{\mathcal{H}^\infty} \leq \|W(D_{p_n} - D_p)D_{k_n}\|_{\mathcal{H}^\infty} + \|WD_p(D_{k_n} - D_k)\|_{\mathcal{H}^\infty}$$

and

$$\left\|(T_{2_n} - T_2)X_{[-\Omega,\Omega]}\right\|_{\mathcal{H}^\infty} \leq \|W(D_{p_n} - D_p)N_{p_n}\|_{\mathcal{H}^\infty} + \left\|WD_p(N_{p_n} - N_p)X_{[-\Omega,\Omega]}\right\|_{\mathcal{H}^\infty}$$

and assumption 8.4.1.

Other results from section 5.5 will be stated when needed. ■

The following theorem captures the main ideas in obtaining a solution to the $\mathcal{H}^\infty$ *Approximate/Design Sensitivity Problem.*

Theorem 6.4.1 (Main Ideas)
Suppose that

$$\lim_{n\to\infty} \mu_n \leq \mu_{opt} \tag{1}$$

and that there exists a uniformly bounded sequence $\{\tilde{Q}_n\}_{n=1}^{\infty} \subset R\mathcal{H}_0^\infty$ *such that*

$$\left\|T_{1_n} - T_{2_n}\tilde{Q}_n\right\|_{\mathcal{H}^\infty} \leq \mu_n + \epsilon, \tag{2}$$

for n sufficiently large, and

$$\lim_{n\to\infty} \left\|1 - \delta(P_n, \tilde{K}_n(\tilde{Q}_n))\right\|_{\mathcal{H}^\infty} = 0. \tag{3}$$

Given the above, $\{\tilde{K}_n\}_{n=1}^{\infty}$ *will internally stabilize* P *with respect to the ring* $\mathcal{H}^\infty$ *for all but a finite number of n. In addition, the actual performance approaches the optimal performance as the approximants get "better"; i.e.*

$$\lim_{n\to\infty} \tilde{\mu}_n = \mu_{opt}.$$

■

Proof

Since

$$\mu_{opt} \stackrel{\text{def}}{=} \inf_{Q \in \mathcal{H}_0^\infty} \left\| \frac{W}{1 - PK(P,Q)} \right\|_{\mathcal{H}^\infty}$$

and

$$\tilde{\mu}_n \stackrel{\text{def}}{=} \left\| \frac{W}{1 - PK(P_n, \tilde{Q}_n)} \right\|_{\mathcal{H}^\infty} = \left\| \frac{(D_p - D_{p_n})(D_{k_n} - N_{p_n}\tilde{Q}_n) + (T_{1_n} - T_{2_n}\tilde{Q}_n)}{\delta(P, \tilde{K}_n(\tilde{Q}_n)} \right\|_{\mathcal{H}^\infty}$$

we have $\mu_{opt} \leq \tilde{\mu}_n$ for each $n \in Z_+$. Consequently,

$$\mu_{opt} \leq \tilde{\mu}_n \leq \frac{\left\| T_{1_n} - T_{2_n}\tilde{Q}_n \right\|_{\mathcal{H}^\infty} + \left\| W(D_p - D_{p_n})(D_{k_n} - N_{p_n}\tilde{Q}_n) \right\|_{\mathcal{H}^\infty}}{1 - \left\| 1 - \delta(P_n, \tilde{K}_n(\tilde{Q}_n)) \right\|_{\mathcal{H}^\infty}}.$$

The result then follows from conditions (1), (2), and (3) within the theorem.

We now show that (3) implies internal stability. To do so, we argue as follows.

Since $\tilde{K}_n$ internally stabilizes P_n, it follows that $N_{k_n} - D_{p_n}\tilde{Q}_n$ and $D_{k_n} - N_{p_n}\tilde{Q}_n$ must be coprime in $\mathcal{H}^\infty$. We also have that N_{p_n} and D_{p_n} are coprime in $\mathcal{H}^\infty$. Consequently, from proposition 3.3.1, we have that $\tilde{K}_n$ internally stabilizes P if and only if $\delta(P, \tilde{K}_n)$ is a unit of (i.e. invertible in) $\mathcal{H}^\infty$. However, (3) implies that $\delta(P, \tilde{K}_n)$ will be a unit for all but a finite number of n. This completes the proof. ■

Theorem 6.4.1 shows precisely what is needed to solve the $\mathcal{H}^\infty$ *Approximate/Design Sensitivity Problem.* It was shown in section 6.3 that one can run into serious difficulty satisfying the third condition of theorem 6.4.1 if one chooses $\tilde{Q}_n = Q_n$. We emphasize that it is condition (3) which will allow us to guarantee internal stability when $\tilde{K}_n$ is used with P. Because of this, the *"traditional" choice of r, as the identity, comes with no guarantees.* In what follows we shall present a way for choosing $\tilde{Q}_n$, and hence the roll-off operator r, so that all three conditions in theorem 6.4.1 are satisfied. With this construction we will have a solution to the $\mathcal{H}^\infty$ *Approximate/Design Sensitivity Problem.*

We shall see that condition (1) in theorem 6.4.1 will follow from the implicit structure of the weighted sensitivity problem. This condition should be interpreted loosely as an "upper-semicontinuity" condition. Such a condition should be expected from the results in [53, pp. 345].

Condition (2) in theorem 6.4.1 will be achieved by exploiting the ideas developed in chapter 5. More specifically, (2) will follow from theorem 5.5.1. We refer to condition (2) as a "sub-optimality" condition.

Finally, condition (3) in theorem 6.4.1 will follow after showing that the $\tilde{Q}_n$ are uniformly bounded and uniformly roll-off in $R\mathcal{H}_0^\infty$. This will be critical in stablishing (3). Since condition (3) gives us stability, we shall refer to it as the "internal stability" condition.

The main result of this section is now stated in the following theorem.

Theorem 6.4.2 (Main Result: Solution to Approximate/Design Problem)

There exists a sequence $\{\tilde{Q}_n\}_{n=1}^\infty \subset R\mathcal{H}_0^\infty$ which is uniformly bounded, uniformly rolls-off. There also exists $N \in Z_+$ such that

$$\mu_{opt} \leq \tilde{\mu}_n \leq \mu_{opt} + \epsilon$$

for all $n \geq N$. Consequently, there exists a roll-off operator $r : Q_n \to \tilde{Q}_n \in R\mathcal{H}_0^\infty$ such that

$$\lim_{n\to\infty} \tilde{\mu}_n = \mu_{opt}.$$

Moreover, the sequence $\{\tilde{Q}_n\}_{n=1}^{\infty}$ generates a sequence of finite dimensional, strictly proper, internally stabilizing compensators $\{\tilde{K}_n\}_{n=1}^{\infty}$, where

$$\tilde{K}_n = \frac{N_{k_n} - D_{p_n}\tilde{Q}_n}{D_{k_n} - N_{p_n}\tilde{Q}_n}$$

and

$$\mu_{opt} \leq \left\| \frac{W}{1 - P\tilde{K}_n} \right\|_{\mathcal{H}^\infty} \leq \mu_{opt} + \epsilon$$

for all $n \geq N$. $\tilde{K}_n$ is thus nearly-optimal for all $n \geq N$.

∎

Proof

To prove the theorem, we only need to show that conditions (1), (2), and (3) of theorem 6.4.1 can be satisfied.

From proposition 5.5.2, we have that

$$\lim_{n\to\infty} \mu_n \leq \mu_{opt}.$$

This gives us the upper-semicontinuity condition (1) in theorem 6.4.1.

From theorem 5.5.1, there exists a sequence $\{\tilde{Q}_n\}_{n=1}^{\infty} \subset R\mathcal{H}_0^\infty$ which is uniformly bounded, uniformly rolls-off. Moreover, there exists $N \in Z_+$ such that

$$\left\| T_{1_n} - T_{2_n}\tilde{Q}_n \right\|_{\mathcal{H}^\infty} \leq \mu_n + \epsilon$$

for each $n \geq N$. This gives us the sub-optimality condition (2) in theorem 6.4.1.

To complete the proof, we only need to show that condition (3) of theorem 6.4.1 holds; i.e.

$$\lim_{n\to\infty} \left\| 1 - \delta(P_n, \tilde{K}_n(\tilde{Q}_n)) \right\|_{\mathcal{H}^\infty} = 0.$$

To do so we note that

$$\delta(P_n, \tilde{K}_n(\tilde{Q}_n)) = D_p(D_{k_n} - N_{p_n}\tilde{Q}_n) - N_p(N_{k_n} - D_{p_n}\tilde{Q}_n)$$

$$= D_p(D_{k_n} - D_k) + D_pD_k - N_p(N_{k_n} - N_k) - N_pN_k - D_p(N_{p_n} - N_p)\tilde{Q}_n - N_p(D_p - D_{p_n})\tilde{Q}_n.$$

Since $D_pD_k - N_pN_k = 1$, this yields

$$1 - \delta(P_n, \tilde{K}_n(\tilde{Q}_n)) = D_p(D_k - D_{k_n}) + N_p(N_{k_n} - N_k) + D_p(N_{p_n} - N_p)\tilde{Q}_n + N_p(D_p - D_{p_n})\tilde{Q}_n.$$

Condition (3) of theorem 6.4.1 then follows from assumption 8.4.1.

Given the above, it follows from theorem 6.4.1 that $\{\tilde{K}_n\}_{n=1}^{\infty}$ will internally stabilize P with respect to the ring $\mathcal{H}^\infty$ for all but a finite number of n. In addition, the *actual performance* approaches the *optimal performance* as the approximants get "better"; i.e.

$$\lim_{n\to\infty} \tilde{\mu}_n = \mu_{opt}.$$

This completes the proof. ∎

Comment 6.4.4 (Internal Stability)
To guarantee internal stability we have used the fact that N_{k_n} uniformly approximates N_k. If N_{k_n} approximated N_k uniformly only on compact frequency intervals, then we would need say, N_p to roll-off in order to guarantee that $\|N_p(N_{k_n} - N_k)\|_{\mathcal{H}^\infty}$ is small for large n.

■

The above theorem shows that given our assumptions on the weighting function, the approximants, and on the plant, there exists a way to construct nearly-optimal, finite dimensional, strictly proper controllers for the infinite dimensional plant; i.e. in a manner such that the *actual performance* approaches the *optimal performance* as the approximants get better. Consequently, we have solved the $\mathcal{H}^\infty$ *Approximate/Design Sensitivity Problem.* It should be emphasized that this has been done by using approximants which converge in the compact topology on $\mathcal{H}^\infty$ and do not necessarily converge in the uniform topology. Moreover, the conditions which the approximants $\{P_n\}_{n=1}^{\infty}$ must satisfy are weak.

Comment 6.4.5 (Direct Construction of Finite Dimensional Compensators)

From theorem 5.2.2, it immediately follows that we can also construct nearly-optimal infinite dimensional compensators for P. In principle, one can approximate such compensators to get finite dimensional compensators. This, however, would defeat our purpose. In our *Approximate /Design* approach we intentionally avoid solving infinite dimensional optimization problems.

To perform the above construction would mean that we would have to solve an infinite dimensional "inner problem" for Z_{opt} (cf. proposition 5.2.1). In the context of this work, however, this is unacceptable.

■

The following examples indicate how the ideas presented in this section can be applied.

Example 6.4.1 (A Stable Plant)

Let our infinite dimensional plant be given by $P = \frac{e^{-s}}{s+1}$. We then have $P = \frac{N_p}{D_p}$ and $D_pD_k - N_pN_k = 1$ where $N_p = \frac{e^{-s}}{s+1}$, $D_p = 1$, $N_k = 0$, and $D_k = 1$.

Let the approximants be given by $P_n = \frac{N_{p_n}}{D_{p_n}}$, where $N_{p_n} = \frac{Pade(n,n)}{s+1}$ and $D_{p_n} = 1$. We then have $D_{p_n}D_{k_n} - N_{p_n}N_{k_n} = 1$ where $N_{k_n} = 0$ and $D_{k_n} = 1$. The construction of near-optimal compensators for P based on P_n, then follows from theorem 6.4.2. Here $Pade(n, n)$ denotes the $[n, n]$ Pade' approximant for e^{-s} (cf. example 2.10.2).

The resulting nearly-optimal compensator takes on the form

$$\tilde{K}_n \stackrel{\text{def}}{=} \frac{-\tilde{Q}_n}{1 - P_n\tilde{Q}_n}$$

where

$$\tilde{Q}_n \stackrel{\text{def}}{=} Z_n\tilde{f}_A g_B.$$

Here, $\tilde{f}_A$ is an $R\mathcal{H}^\infty$ function which is sufficiently close to $f_A \stackrel{\text{def}}{=} (\frac{1}{s+1})^{\frac{1}{A}} \in \mathcal{H}_0^\infty$, where $A \stackrel{\text{def}}{=} A(\epsilon, W) \in Z_+$ is sufficiently large and fixed. Also, $g_B \stackrel{\text{def}}{=} \frac{s+1}{s+\frac{1}{B}} \in R\mathcal{H}^\infty$, where $B \stackrel{\text{def}}{=} B(\epsilon, A) \in Z_+$ is sufficiently large and fixed. How $\tilde{f}_A$ and g_B are precisely constructed, is shown in theorem 5.5.1.

The above, compensator is finite dimensional, strictly proper, and will be nearly-optimal for n sufficently large. How large n needs to be can be determined from theorem 5.5.1 and the results of this section.

$\blacksquare$

Example 6.4.2 (An Unstable Plant)

Let our infinite dimensional plant be given by $P = \frac{e^{-s}}{s-1}$. Also let $P = \frac{N_p}{D_p}^{-1}$, where $N_p = \frac{e^{-s}}{s+1}$ and $D_p = \frac{s-1}{s+1}$.

If we select the nominal compensator $K_{nom} \stackrel{\text{def}}{=} N_k D_k^{-1}$, where $N_k = -2e$ and $D_k = \frac{s+1-2e^{1-s}}{s-1}$, then we obtain $D_p D_k - N_p N_k = 1$. We thus have a coprime factorization (N_p, D_p) for P over $\mathcal{H}^\infty$ with Bezout factors (N_k, D_k).

The problem here is that N_k, and hence K_{nom}, does not roll-off (cf. comment 6.4.3). We need to construct a nominal compensator $K_{nom} \stackrel{\text{def}}{=} N_k D_k^{-1}$ which is strictly proper. Such a compensator will be found using classical ideas.

We shall obtain the desired compensator based on the "design plant" $\frac{1}{s-1}$. Let

$$K_{nom} \stackrel{\text{def}}{=} M \, \frac{s+a}{(s+b)(s+c)}$$

where a, b, c, M are design parameters. It should be clear that there exists $a, b, c, M \in R_+$ such that $K_{nom}(s)$ internally stabilizes P. To see this one first uses root locus ideas to see that $K_{nom}(s)$ will stabilize the "design plant" $\frac{1}{s-1}$. The idea, then is to adjust the parameters so that the "design loop", determined by $\frac{1}{s-1}$ and $K_{nom}(s)$, remains stable even in the presence of a loop perturbation e^{-s}. This can be done by appropriately selecting the gain crossover frequency and the phase margin of the "design loop". One can choose, for example, $a = 1$, $b = 1000$, $c = 10b$ and then adjust M appropriately.

We now obtain a coprime factorization (N_k, D_k) for K_{nom} over $\mathcal{H}^\infty$ with Bezout factors (N_p, D_p). To do so, we first consider the loop determined by P and K_{nom}. It has characteristic "polynomial" given by

$$d(s) \stackrel{\text{def}}{=} (s-1)\,(s+b)\,(s+c) \;-\; M\,e^{-s}\,(s+a).$$

By construction, all of its roots lie in the open left half plane. We now define

$$\tilde{N}_k \stackrel{\text{def}}{=} K_{nom}$$

$$\tilde{D}_d \stackrel{\text{def}}{=} 1$$

and

$$\Phi \stackrel{\text{def}}{=} D_p \tilde{D}_k - N_p \tilde{N}_k = \frac{d(s)}{(s+1)(s+b)(s+c)}.$$

By construction $\Phi \in \mathcal{H}^\infty$. Also, since $d(s)$ is stable and Φ is proper, we conclude that Φ is a unit of $\mathcal{H}^\infty$. Given this, if we define

$$N_k \stackrel{\text{def}}{=} \frac{\tilde{N}_k}{\Phi}$$

and

$$D_k \stackrel{\text{def}}{=} \frac{\tilde{D}_k}{\Phi},$$

then it follows that $D_pD_k - N_pN_k = 1$. Moreover, $N_k, D_k \in \mathcal{C}_e$ and hence by proposition 2.10.1 can be uniformly approximated by $R\mathcal{H}^\infty$ functions. It thus follows that N_p, D_p, N_k, and D_k satisfy the needed conditions.

Given the above, we then let our approximants be given by $P_n = \frac{N_{p_n}}{D_{p_n}}$, where $N_{p_n} = \frac{Pade(n,n)}{s+1}$, $D_{p_n} = \frac{s-1}{s+1}$. Moreover, we let $N_{k_n} \in R\mathcal{H}_0^\infty$ and $D_{k_n} \in R\mathcal{H}^\infty$ be approximations of N_k and D_k, respectively. The construction of near-optimal compensators for P based on P_n, then follows from theorem 6.4.2. Here $Pade(n,n)$ denotes the $[n,n]$ Pade' approximant for e^{-s} (cf. example 2.10.2).

The resulting nearly-optimal compensator takes on the form

$$\tilde{K}_n \stackrel{\text{def}}{=} \frac{N_{k_n} - D_{p_n}\tilde{Q}_n}{D_{k_n} - N_{p_n}\tilde{Q}_n}$$

where the construction of $\tilde{Q}_n$ is performed in a manner which is analogous to that discussed in the previous example.

■

6.5 Solution to $\mathcal{H}^\infty$ Purely Finite Dimensional Sensitivity Problem: Computation of Optimal Performance

In practice we often would like to compute or estimate the *optimal performance* μ_{opt}. The following theorem says that under our assumptions, the *expected performance* μ_n approaches the *optimal performance* μ_{opt}.

Theorem 6.5.1 (Solution to Purely Finite Dimensional Problem)

$$\lim_{n\to\infty} \mu_n = \mu_{opt}.$$

■

This solves the *Purely Finite Dimensional Sensitivity Problem.*

Proof The proof of this follows from that of theorem 5.5.2. ■

It should be emphasized that the above result has been obtained even though the *optimal performance* μ_{opt} need not be continuous with respect to perturbations in the plant P, even when the uniform topology is imposed [53]. Even if it were continuous, there are many plants in $\mathcal{H}^\infty$ which cannot be approximated by $R\mathcal{H}^\infty$ approximants; e.g. a delay (see proposition 2.10.1). Given this, it is also imperative to point out that the above result has been shown even though the approximants $\{P_n\}_{n=1}^\infty$ need not approximate the plant P uniformly.

The approach usually taken in the literature to compute μ_{opt} is to solve an infinite dimensional eigenvalue/eigenfunction problem [14, pp. 28–31], [61], [63, pp. 308]. Theorem 6.5.1, however implies that in order to estimate μ_{opt} all we need do is solve a sequence of finite dimensional eigenvalue/eigenvector problems. This computational virtue is better exhibited in the following "spectral" corollary to theorem 6.5.1, which shows explicitly the relationship between the relevant finite dimensional eigenvalue/eigenvector problems and the infinite dimensional eigenvalue/eigenfunction problem.

Corollary 6.5.1 (Spectral Implication)
If

$$\max\{\ |T_1(j\infty)|,\ \max_k |T_1(j\omega_k)|\ \} \le \left\|\Gamma_{T_1 T_{2_i}^*}\right\|,$$

then

$$\lim_{n\to\infty} \left\|\Gamma_{T_1 T_{2_i}^*}\right\| = \mu_{opt}.$$

■

Proof From proposition 5.3.1, it follows that

$$\mu_{opt} = \max\{|T_1(j\infty)|,\ \max_k |T_1(j\omega_k)|,\ \left\|\Gamma_{T_1 T_{2_i}^*}\right\|\ \}.$$

From the assumption, we have $\mu_{opt} = \left\|\Gamma_{T_1 T_{2_i}^*}\right\|$. The result then follows immediately from theorem 5.4.1. ■

Comment 6.5.1 (Hankel Norm Assumption)
The inequality in the above coroallary is satisfied if the points ∞, $\{\omega_k\}$ are essential singularities of T_{2_i}. This has been shown in [61]. Without going into detail, this is because in such a case $|T_1(j\infty)|$ and $\{|T_1(j\omega_k)|\}$ lie in the *essential spectrum* of the Hankel operator $\Gamma_{T_1 T_{2_i}^*}$ and essential spectra cannot exceed the operator norm. If, for example, $T_2 = e^{-s}$, then the inequality is satisfied.

We note that lemma 5.3.1 may help in choosing T_1 so that the Hankel norm assumption in corollary 6.5.1 is satisfied (cf. [14, chapter 8]).

■

This corollary shows that given our assumptions, the above norms converge. We emphasize here that we have proven convergence of the norms even though the finite rank Hankel operators $\{\Gamma_{T_{1_n} T_{2_{n_i}}^*}\}_{n=1}^{\infty}$ do not, in general, converge uniformly to the Hankel operator $\Gamma_{T_1 T_{2_i}^*}$. This is because $\Gamma_{T_1 T_{2_i}^*}$ is usually non-compact and thus cannot be approximated uniformly by a sequence of finite rank operators [5, pp. 42, 178]. The Hankel operator $\Gamma_{T_1 T_{2_i}^*}$ can be shown to be non-compact, for example, when the weighting W is only proper and the plant P is a pure delay ($T_1 = W$, $T_{2_i} = e^{-s}$). This is seen from Hartman's Theorem on compact Hankel operators (proposition 2.9.2). Consequently, our approach also avoids finding approximate solutions via Hankel operator approximation theory which falls apart for non-compact operators.

Although the above provides a solution to the *Purely Finite Dimensional Sensitivity Problem*, it does not really provide insight into the computation of the *optimal performance* μ_{opt}. Such insight can be obtained from the results in sections 5.4 and 5.5.

Lets assume that $P \in \mathcal{H}^\infty$. In practice one might compute the approximants P_n using Pade' approximations [3], for example. Such approximants were presented in example 2.10.2 for the case where P was a delay.

The following examples illustrate how such approximants can be used to solve problems which would ordinarily require the solution of an infinite dimensional eigenvalue/eigenfunction problem.

Example 6.5.1 (Tannenbaum)
In this example we consider a problem from [18]. Our infinite dimensional plant is a delay: $P = e^{-s\Delta}$

where $\Delta = 1$. The weighting function is low-pass and given by $W(s) = \frac{1}{as+1}$ with $a = 100$. To compute $\|\Gamma_{WP_i^*}\|$ we first solve

$$\tan(y) + \frac{ya}{\Delta} = 0$$

for $y_o = 1.577137$. We then have

$$\|\Gamma_{WP_i^*}\| = |W(j\frac{y_o}{\Delta})| = 0.00634.$$

Now let $P_n = [n, n]$ Pade' approximant for P (cf. example 2.10.2). Given this, one can show that

$$\{\left\|\Gamma_{WP_{n_i}^*}\right\|\}_{n=1}^{\infty} = \{0.00498, 0.00629, 0.00634, \ldots\}.$$

Since $|W(j\infty)| = 0 \le \|\Gamma_{WP_i^*}\|$, we expect $\lim_{n\to\infty} \|\Gamma_{WP_{ni}^*}\| = \|\Gamma_{WP_i^*}\|$ from theorem 5.4.1. The computer data shows that we have convergence as predicted. One might argue that this case is easy since Hankel operator is compact.

∎

Example 6.5.2 (Flamm, Mitter)
In this example we consider a problem from [14], [15]. Our infinite dimensional plant is a delay: $P = e^{-s\Delta}$ where $\Delta = 1$. The weighting function is low-pass and given by $W(s) = \frac{s+1}{s+\beta}$ with $\beta = 0.01$. To compute $\|\Gamma_{WP_i^*}\|$ we first solve

$$\cot(\omega\Delta) = \frac{\omega^2 - \beta}{\omega(1+\beta)}$$

for $\omega_o = 0.8676622$. We then have

$$\|\Gamma_{WP_i^*}\| = |W(j\omega_o)| = 1.5258.$$

Now let $P_n = [n, n]$ Pade' approximant for P(cf. example 2.10.2). Given this, one can show that

$$\{\left\|\Gamma_{WP_{n_i}^*}\right\|\}_{n=1}^{\infty} = \{1.4925, 1.5254, 1.5258, \ldots\}.$$

Since $|W(j\infty)| = \inf_{\omega\in R_e} |W(j\omega)|$, it follows form lemma 5.3.1 that $|W(j\infty)| \le \|\Gamma_{WP_i^*}\|$. Given this, we expect $\lim_{n\to\infty} \|\Gamma_{WP_{ni}^*}\| = \|\Gamma_{WP_i^*}\|$ from (4) of theorem 5.4.1 and the fact that P_n uniformly approximates P on compact intervals (see example 2.10.2). Although the Hankel operator $\Gamma_{WP_i^*}$ is non-compact, the computer data shows that we have convergence as predicted.

∎

6.6 Solution to $\mathcal{H}^\infty$ Loop Convergence Sensitivity Problem

Thus far, we have given conditions under which the *expected performance* μ_n and the *actual performance* $\tilde{\mu}_n$ approach the *optimal performance* μ_{opt}. We now investigate the behavior of the actual loop shapes. More specifically, we would like to know in what sense, if any, does $\tilde{Q}_n$ converge to a near-optimal solution $\tilde{Q}_{opt}$. Since $\tilde{Q}_n$ and $\tilde{Q}_{opt}$ are based on "inner solutions" Z_n and Z_{opt}, it makes sense to investigate the convergence of Z_n to Z_{opt}. We start with the following assumption.

Assumption 6.6.1 (Uniform Convergence)
We shall further assume that

$$\lim_{n\to\infty} \left\| T_{2_{n_i}} - T_{2_i} \right\|_{\mathcal{H}^\infty} = 0.$$

■

Comment 6.6.1 (Applicability)
The above assumption will not be satified for a large class of plants; e.g. $P = e^{-s}$. This is because $R\mathcal{H}^\infty$ functions are not dense in $\mathcal{H}^\infty$. A delay, for example, cannot be uniformly approximated by $R\mathcal{H}^\infty$ functions (cf. proposition 2.10.1). The assumption can be satisfied if, for example, $T_{2_i} \in \mathcal{C}_e$; i.e. real-rational (cf. proposition 2.7.3). Weakening the above assumption will be a topic for future research.

■

Given the above assumption, we have the following proposition.

Proposition 6.6.1 (Weak* Loop Convergence)
Let $Z_{opt} \in \mathcal{H}^\infty$ be the unique solution to

$$\min_{Z\in\mathcal{H}^\infty} \|T_1 - T_{2_i} Z\|_{\mathcal{H}^\infty} = \|T_1 - T_{2_i} Z_{opt}\|_{\mathcal{H}^\infty}.$$

It then follows that $\{Z_n\}_{n=1}^{\infty}$ converges to Z_{opt} in the weak* topology on $\mathcal{H}^\infty$.

■

Proof

We prove the proposition, we begin by noting that $\mathcal{H}^\infty$ is the dual space of the *quotient space* $\mathcal{L}^1/\mathcal{H}^1$ [30, pp. 137]. Since it is a dual space, its elements may be viewed as bounded linear functionals acting on the primal space $\mathcal{L}^1/\mathcal{H}^1$. Given this, we recall Alaoglu's theorem which says that the closed unit ball in a dual space is weak* compact (cf. proposition 2.11.4).

Recall that $Z_n \in R\mathcal{H}^\infty$ is a solution to the finite dimensional inner problem

$$\min_{Z\in R\mathcal{H}^\infty} \left\| T_{1_n} - T_{2_{n_i}} Z \right\|_{\mathcal{H}^\infty} = \left\| T_{1_n} - T_{2_{n_i}} Z_n \right\|_{\mathcal{H}^\infty}.$$

Since T_{1_n} is uniformly bounded, so is Z_n. This is because

$$\|Z_n\|_{\mathcal{H}^\infty} \le \min_{Z\in R\mathcal{H}^\infty} \left\| T_{1_n} - T_{2_{n_i}} Z \right\|_{\mathcal{H}^\infty} + \|T_{1_n}\|_{\mathcal{H}^\infty} \le 2\|T_{1_n}\|_{\mathcal{H}^\infty}.$$

From Alaoglu's theorem, it thus follows that Z_n possesses a subsequence $\{Z_{n(k)}\}_{k=1}^{\infty}$ which converges in the weak* topology on $\mathcal{H}^\infty$ to say $L \in \mathcal{H}^\infty$. Since $T_{2_i}(\mathcal{L}^1/\mathcal{H}^1) \subset \mathcal{L}^1/\mathcal{H}^1$, this implies that $T_1 - T_{2_i} Z_{n(k)}$ is weak* convergent to $T_1 - T_{2_i} L$. Proposition 2.11.1 thus gives us the following inequality:

$$\|T_1 - T_{2_i} L\|_{\mathcal{H}^\infty} \le \lim_{k\to\infty} \left\| T_1 - T_{2_i} Z_{n(k)} \right\|_{\mathcal{H}^\infty}.$$

From this inequality, we obtain

$$\|T_1 - T_{2_i} L\|_{\mathcal{H}^\infty} \le \lim_{k\to\infty} \{ \left\| T_{1_{n(k)}} - T_{2_{n(k)_i}} Z_{n(k)} \right\|_{\mathcal{H}^\infty} + \left\| T_1 - T_{1_{n(k)}} \right\|_{\mathcal{H}^\infty} + \left\| (T_{2_i} - T_{2_{n(k)_i}}) Z_n \right\|_{\mathcal{H}^\infty} \}.$$

Using assumption 6.6.1 allows the last two terms to vanish. From proposition 5.2.1 it then follows that

$$\|T_1 - T_{2_i} L\|_{\mathcal{H}^\infty} \leq \lim_{k \to \infty} \left\| \Gamma_{T_{1_{n(k)}} T^*_{2_{n(k)_i}}} \right\|.$$

From assumption 6.6.1 and theorem 5.4.1 it follows that the right hand side is equal to $\left\| \Gamma_{T_1 T^*_{2_i}} \right\|$. From proposition 5.2.1, it thus follows that

$$\|T_1 - T_{2_i} L\|_{\mathcal{H}^\infty} \leq \left\| \Gamma_{T_1 T^*_{2_i}} \right\| = \|T_1 - T_{2_i} Z_{opt}\|_{\mathcal{H}^\infty} = \min_{Z \in \mathcal{H}^\infty} \|T_1 - T_{2_i} Z\|_{\mathcal{H}^\infty}.$$

Since, by assumption, Z_{opt} is the unique minimum, the above inequality implies that $L = Z_{opt}$. Moreover, we have also shown that any weak* convergent subsequence of Z_n necessarily converges to Z_{opt}. The result then follows from proposition 2.11.5. ∎

Theorem 6.6.1 (Solution to Loop Convergence Problem)
The sequence $\{Z_n\}_{n=1}^{\infty}$ converges uniformly to Z_{opt} on all compact frequency intervals.

∎

Proof
The sequence $\{Z_n\}_{n=1}^{\infty}$ is uniformly bounded in $\mathcal{H}^\infty$. By the Arzela-Ascoli theorem (cf. proposition 2.12.1), the sequence $\{Z_n\}_{n=1}^{\infty}$ constitutes a normal family of analytic functions over the open right half plane. This implies that there exists a subsequence $\{Z_{n(k)}\}_{k=1}^{\infty}$ which converges uniformly to say M on all compact subsets of the open right half plane. From proposition 2.10.2, it follows that M is analytic in the open right half plane. Since Z_n is uniformly bounded in $\mathcal{H}^\infty$, it then follows that $M \in \mathcal{H}^\infty$. Using limiting arguments, one can show that uniform convergence on compact subsets of the open right half plane implies uniform convergence on all compact frequency intervals of the imaginary axis.

Lets view $Z_{n(k)}$ and M as bounded linear functionals acting on the primal space $\mathcal{L}^1/\mathcal{H}^1$. Using the uniform convergence on compact frequency intervals and the fact that functions in $\mathcal{L}^1/\mathcal{H}^1$ roll-off, gives us that $Z_{n(k)}$ has weak* limit M. From proposition 6.6.1, it follows that $M = Z_{opt}$. Consequently, $Z_{n(k)}$ converges to Z_{opt} on compact frequency intervals.

The above shows that any compactly convergent subsequence $\{Z_{n(k)}\}_{k=1}^{\infty}$ must converge to Z_{opt}.

Suppose that Z_n does not converge compactly to Z_{opt}. Then, there exists a subsequence $\{Z_{n(k)}\}_{k=1}^{\infty}$ and a compact set S such that

$$\left\| (Z_{n(k)} - Z_{opt}) X_S \right\|_{\mathcal{H}^\infty} \geq \epsilon$$

for all k. This, however, contradicts the normality of $\{Z_n\}_{n=1}^{\infty}$. It must therefore be that Z_n converges to Z_{opt} uniformly on all compact frequency intervals. ∎

Comment 6.6.2 (Convergence of Loop Shapes)
The above theorem implies that if proper care is taken, we can construct sequences of compensators which yield loop shapes which approach the optimal loop shape uniformly on compact subsets. The practical implications of this are obvious. This makes the Approximate/Design approach presented in this chapter a truely useful design tool.

∎

This concludes our discussion of the $\mathcal{H}^\infty$ Loop Convergence Sensitivity Problem.

6.7 Summary

In this chapter a solution was presented to the $\mathcal{H}^\infty$ *Approximate/Design Sensitivity Problem.* More specifically, it was shown how near-optimal finite dimensional strictly proper compensators could be constructed for an infinite dimensional plant, given a weighted $\mathcal{H}^\infty$ sensitivity design specification. It was shown that the construction could be carried out on the basis of finite dimensional solutions obtained from appropriately formulated finite dimensional $\mathcal{H}^\infty$ sensitivity problems. The finite dimensional problems which one solves are "natural" weighted $\mathcal{H}^\infty$ sensitivity problems obtained by replacing the infinite dimensional plant by "appropriately" chosen finite dimensional approximants.

It was shown that, in general, approximants based on "open loop intuition", rather than on the control objective, may yield compensators which do not even guarantee stability when used with the infinite dimensional plant. It was also shown how "appropriate" approximants could be constructed. The approximants obtained were constructed so that their imaginary axis pole-zero structure would not drastically differ from that of the plant. The construction presented does not require sophisticated mathematics or software. It can be used by practicing engineers with little effort.

We also provided a solution to the *Purely Finite Dimensional* $\mathcal{H}^\infty$ *Sensitivity Problem.* Here, the issue of computing the optimal performance was addressed. It was shown that the optimal performance could be computed by solving a sequence of finite dimensional eigenvalue/eigenvector problems rather than the typical infinite dimensional eigenvalue/eigenfunction problems which appear in the literature. This makes a once difficult problem, almost trivial. Examples were given to illustrate this.

Finally, conditions were given under which the near-optimal finite dimensional loop shapes could converge to the optimal infinite dimensional loop shape uniformly on compact frequency intervals. This was illustrated in our solution to the $\mathcal{H}^\infty$ *Loop Convergence Sensitivity Problem.* This makes the Approximate/Design approach presented in the chapter a truely promising engineering tool.

Chapter 7

Design via $\mathcal{H}^\infty$ Mixed-Sensitivity Optimization

7.1 Introduction

In this chapter we consider the problem of designing near-optimal finite dimensional compensators for infinite dimensional plants via $\mathcal{H}^\infty$ mixed-sensitivity optimization. Such an approach can be motivated by design specifications which require some specified degree of robustness or $\mathcal{L}^2$ disturbance rejection. A systematic procedure is presented. More specifcally, we provide a solution to the $\mathcal{H}^\infty$ *Approximate/Design Mixed-Sensitivity Problem*, the $\mathcal{H}^\infty$ *Purely Finite Dimensional Mixed-Sensitivity Problem*, and the $\mathcal{H}^\infty$ *Loop Convergence Mixed-Sensitivity Problem.* Since the ideas are similar to those presented in the previous chapter, we shall focus on stable plants.

7.2 $\mathcal{H}^\infty$ Approximate/Design Mixed-Sensitivity Problem

In this section we present some definitions and assumptions to precisely state the $\mathcal{H}^\infty$ *Approximate/Design Mixed-Sensitivity Problem.* Notation to be used throughout the chapter is also established.

Since we assume that our infinite dimensional plant is stable we have that $P \in \mathcal{H}^\infty$. Given this, it follows from proposition 3.2.1 that the set of all compensators which internally stabilize P, with respect to the ring $\mathcal{H}^\infty$, are parameterized by

$$K(P,Q) \stackrel{\text{def}}{=} \frac{-Q}{1-PQ}$$

where Q is any element in $\mathcal{H}^\infty$. From this, it follows that if we allow Q to vary over $\mathcal{H}_0^\infty$, then we get all strictly proper compensators which internally stabilize P. We shall be doing this throughout the chapter; i.e. all infimizations involving P shall be carried out over $\mathcal{H}_0^\infty$.

In this chapter we shall consrtruct $R\mathcal{H}^\infty$ approximants $\{P_n\}_{n=1}^\infty$ for P. From proposition 3.2.1, it follows that the set of all compensators which internally stabilize P_n, with respect to the ring $\mathcal{H}^\infty$, are parameterized by

$$K(P_n,Q) \stackrel{\text{def}}{=} \frac{-Q}{1-P_nQ}$$

where Q is any element in $\mathcal{H}^\infty$. From this, it follows that if we allow Q to vary over $\mathcal{H}_0^\infty$, then we get all strictly proper compensators which internally stabilize P_n. We shall be doing this throughout the chapter; i.e. all infimizations involving P_n shall be carried out over $\mathcal{H}_0^\infty$.

In this section we shall formulate an $\mathcal{H}^\infty$ mixed-sensitivity problem. To do so, we shall need two frequency dependent *weighting functions*, W_1 and W_2. The following "standard" assumption, on the weighting functions W_1 and W_2, shall be made throughout the chapter.

Assumption 7.2.1 (Weighting Functions)

(1) $W_1 \in R\mathcal{H}^\infty$ *and minimum phase.*
(2) $W_2, W_2^{-1} \in R\mathcal{H}^\infty$.

∎

The above implies that W_1 and W_2 have no poles or zeros in the closed right half plane. W_1 may or may not roll-off.

Suppose our mixed-sensitivity criterion is based a performance measure

$$J(P, K(P,Q)) \stackrel{\text{def}}{=} \left\| \frac{\begin{pmatrix} W_1 \\ W_2 PK(P,Q) \end{pmatrix}}{1 - PK(P,Q)} \right\|_{\mathcal{H}^\infty} = \left\| \begin{pmatrix} W_1(1-PQ) \\ W_2 PQ \end{pmatrix} \right\|_{\mathcal{H}^\infty}$$

which penalizes the sensitivity and complementary sensitivity transfer functions. Here Q varies over $\mathcal{H}_0^\infty$ since we are interested only in strictly proper compensators. It is easy to see that such a criterion possesses pathologies similar to those possessed by the $\mathcal{H}^\infty$ sensitivity problem. We note, for example, that such a criterion will do its best to invert the outer part of P. The criterion, will thus, in general generate improper compensators. Although this difficulty, as will become apparent, can be addressed using the techniques presented in the previous chapters, it can be altogether avoided.

In this chapter we shall present new results for a mixed-sensitivity criterion which, in our opinion, is a "better" and more "natural" criterion. The criterion we shall consider is one which penalizes the transfer function associated with the control, as well as the sensitivity function. By doing so we will be able to obtain proper compensators to start with. Obtaining a suboptimal finite dimensional compensator which rolls-off then becomes the central issue. We shall see that to achieve this objective, all one need do is apply the ideas presented in Chapter 5 for the sensitivity problem. To provide a logical flow of ideas we proceed as we did for the sensitivity problem in Chapter 6.

To begin our development, we define the $\mathcal{H}^\infty$ *mixed-sensitivity measure* to be used throughout the chapter as follows.

Definition 7.2.1 ($\mathcal{H}^\infty$ Mixed-Sensitivity Measure)
Let $F, G, Q \in \mathcal{H}^\infty$. Also let $K(G,Q)$ denote a compensator which internally stabilizes G with respect to the ring $\mathcal{H}^\infty$. If it also internally stabilizes F, it is appropriate to define the $\mathcal{H}^\infty$ mixed-sensitivity measure of the pair $(F, K(G,Q))$ as follows:

$$J(F, K(G,Q)) \stackrel{\text{def}}{=} \left\| \frac{\begin{pmatrix} W_1 \\ W_2 K(G,Q) \end{pmatrix}}{1 - FK(G,Q)} \right\|_{\mathcal{H}^\infty}$$

From proposition 3.2.1 it follows that $K(G,Q) = \frac{-Q}{1-PQ}$. *We thus have*

$$J(F, K(G,Q)) = \left\| \frac{\begin{pmatrix} W_1(1-GQ) \\ W_2Q \end{pmatrix}}{1-(G-F)Q} \right\|_{\mathcal{H}^\infty}.$$

■

The above shows that when $\|(F-G)Q\|_{\mathcal{H}^\infty} < 1$, then $K(G,Q)$ "internally" stabilizes F and J is well defined. This follows from the *Small Gain Theorem* [11]. We also see that if $F = G$, then this "small gain" assumption is automatically satisfied, and hence J is well defined. This is because $K(F,Q)$ "internally"stabilizes F for any $Q \in \mathcal{H}^\infty$, by definition. These issues shall receive further consideration below.

From definition 7.2.1 above, it follows that $J(P, K(P,Q)) = \left\| \begin{pmatrix} W_1(1-PQ) \\ W_2Q \end{pmatrix} \right\|_{\mathcal{H}^\infty}$. Substituting into definition 5.2.1 and allowing Q to vary over $\mathcal{H}_0^\infty$, then gives us the following expression for the *optimal performance* μ_{opt}.

Definition 7.2.2 (Optimal Performance)

$$\mu_{opt} \stackrel{\text{def}}{=} \inf_{Q \in \mathcal{H}_0^\infty} \left\| \frac{\begin{pmatrix} W_1 \\ W_2K(P,Q) \end{pmatrix}}{1-PK(P,Q)} \right\|_{\mathcal{H}^\infty} = \inf_{Q \in \mathcal{H}_0^\infty} \left\| \begin{pmatrix} W_1(1-PQ) \\ W_2Q \end{pmatrix} \right\|_{\mathcal{H}^\infty}.$$

■

We emphasize that this definition defines an infinite dimensional optimization problem; one which we want to, and will, avoid solving. We denote its infimizer by $Q_{opt} \in \mathcal{H}^\infty$ [1].

Similarly, from definition 7.2.1 above, it follows that $J(P, K(P,Q)) = \left\| \begin{pmatrix} W_1(1-PQ) \\ W_2Q \end{pmatrix} \right\|_{\mathcal{H}^\infty}$. After substituting this into definition 5.2.1, and allowing Q to vary over $R\mathcal{H}_0^\infty$, one obtains the following expression for the *expected performance* μ_n.

Definition 7.2.3 (Expected Performance)

$$\mu_n \stackrel{\text{def}}{=} \inf_{Q \in R\mathcal{H}_0^\infty} \left\| \frac{\begin{pmatrix} W_1 \\ W_2K(P_n,Q) \end{pmatrix}}{1-P_nK(P_n,Q)} \right\|_{\mathcal{H}^\infty} = \inf_{Q \in R\mathcal{H}_0^\infty} \left\| \begin{pmatrix} W_1(1-P_nQ) \\ W_2Q \end{pmatrix} \right\|_{\mathcal{H}^\infty}.$$

■

Here we can infimize over $R\mathcal{H}_0^\infty$ since W_1, W_2, and P_n are real-rational [23].

Let Q_n achieve the infimum (or be near-optimal) in definition 8.2.3. We note that in general Q_n will lie in $\mathcal{H}^\infty$. This is because W_2 is invertible in $\mathcal{H}^\infty$. Consequently, Q_n will generate a finite dimensional compensator

$$K_n \stackrel{\text{def}}{=} K(P_n, Q_n) = \frac{-Q_n}{1-P_nQ_n}$$

which generally does not roll-off. Also K_n will not necessarily stabilize the infinite dimensional plant P (cf. example 7.3.1).

[1]Existence issue ???

In an attempt to remedy these shortcomings, we define a *roll-off operator*

$$r : Q_n \rightarrow \tilde{Q}_n \in R\mathcal{H}_0^\infty.$$

The exact form of r, as we shall see, is obtained in a manner analogous to that used for the sensitivity problem. To construct r all that will be required are the ideas in theorem 5.2.1. The exact form of r will be determined subsequently.

The compensator generated by $\tilde{Q}_n$ is given by

$$\tilde{K}_n \stackrel{\text{def}}{=} K(P_n, \tilde{Q}_n) = \frac{-\tilde{Q}_n}{1 - P_n\tilde{Q}_n}.$$

Given this, we consider the feedback system obtained by substituting $\tilde{K}_n$ into a closed loop system with the infinite dimensional plant P. Let $H(P, \tilde{K}_n)$ denote the resulting closed loop tranfser function matrix from r, d to e, u. We then have

$$\begin{bmatrix} e \\ u \end{bmatrix} = H(P, \tilde{K}_n) = \begin{bmatrix} r \\ d \end{bmatrix},$$

where

$$H(P, \tilde{K}_n) = \begin{bmatrix} \frac{1}{1-P\tilde{K}_n} & \frac{P}{1-P\tilde{K}_n} \\ \frac{\tilde{K}_n}{1-P\tilde{K}_n} & \frac{1}{1-P\tilde{K}_n} \end{bmatrix}.$$

Substituting for $\tilde{K}_n$, then gives

$$H(P, \tilde{K}_n(\tilde{Q}_n)) = \begin{bmatrix} 1 - P_n\tilde{Q}_n & P(1 - P_n\tilde{Q}_n) \\ -\tilde{Q}_n & 1 - P_n\tilde{Q}_n \end{bmatrix} \frac{1}{1 - (P_n - P)\tilde{Q}}.$$

Given that internal stability can be shown, the *actual performance* $\tilde{\mu}_n$ defined in definition 4.3.3 is well defined and becomes:

Definition 7.2.4 (Actual Performance)

$$\tilde{\mu}_n \stackrel{\text{def}}{=} \left\| \frac{\begin{pmatrix} W_1 \\ W_2 K(P_n, \tilde{Q}_n) \end{pmatrix}}{1 - PK(P_n, \tilde{Q}_n)} \right\|_{\mathcal{H}^\infty} = \left\| \frac{\begin{pmatrix} W_1(1 - P_n\tilde{Q}_n) \\ W_2\tilde{Q}_n \end{pmatrix}}{1 - (P_n - P)\tilde{Q}_n} \right\|_{\mathcal{H}^\infty}.$$

∎

Given the above definitions, the $\mathcal{H}^\infty$ *Approximate/Design Mixed-Sensitivity Problem* then becomes to find conditions on the approximants $\{P_n\}_{n=1}^\infty$, and on the *roll-off operator* r, such that the *actual performance* approaches the *optimal performance*; i.e.

$$\lim_{n\to\infty} \mu_n = \mu_{opt}.$$

Equivalently, this problem can be viewed as that of finding a near-optimal compensator for the infinite dimensional plant P. The problem also addresses the question: What is a "good" finite dimensional approximant?

Because this problem is of primary concern in this chapter, we now indicate what difficulties are associated with the problem.

7.3 Why is the Approximate/Design Problem Hard ?

The idea behind the *Approximate/Design* approach is that if P_n is close to P, then $\tilde{\mu}_n$ should be close to μ_{opt}. One also might expect μ_n to be close to μ_{opt}. This is not true. The following example shows that even in simple situations we may have a discontinuity.

Example 7.3.1 (Discontinuity)

Let $P = \frac{1}{s+1}$ be our plant. Suppose $\|W_2\|_{\mathcal{H}^\infty} < |W_1(j\infty)|$. Also suppose that

$$\mu_{opt} \stackrel{\text{def}}{=} \inf_{Q\in\mathcal{H}^\infty} \left\| \frac{\begin{pmatrix} W_1 \\ W_2PK(P,Q) \end{pmatrix}}{1-PK(P,Q)} \right\|_{\mathcal{H}^\infty} = \inf_{Q\in\mathcal{H}^\infty} \left\| \begin{pmatrix} W_1(1-PQ) \\ W_2PQ \end{pmatrix} \right\|_{\mathcal{H}^\infty}.$$

Here we penalize the sensitivity and complementary sensitivity functions. We now show that μ_{opt} is discontinuous at P. We first note that

$$\mu_{opt} \geq |W_1(j\infty)| > \|W_2\|_{\mathcal{H}^\infty}.$$

Let $P_n = \frac{\frac{s}{n}+1}{s+1}$ approximate P. We see that P_n approximates P uniformly in $\mathcal{H}^\infty$. Also let

$$\mu_n \stackrel{\text{def}}{=} \inf_{Q\in R\mathcal{H}^\infty} \left\| \frac{\begin{pmatrix} W_1 \\ W_2P_nK(P_n,Q) \end{pmatrix}}{1-P_nK(P_n,Q)} \right\|_{\mathcal{H}^\infty} = \inf_{Q\in R\mathcal{H}^\infty} \left\| \begin{pmatrix} W_1(1-P_nQ) \\ W_2P_nQ \end{pmatrix} \right\|_{\mathcal{H}^\infty}.$$

We see that choosing $Q = \frac{1}{P_n} \in \mathcal{H}^\infty$ shows that

$$\mu_n \leq \|W_2\|_{\mathcal{H}^\infty} < |W_1(j\infty)| \leq \mu_{opt}.$$

Consequently, μ_n cannot approach μ_{opt}, as n approaches infinity, even though P_n does approach P. We thus have a discontinuity at P.

∎

The above example shows that when the sensitivity and complementary sensitivity are penalized, then the resulting optimization problem exhibits pathologies identical to those possessed by the $\mathcal{H}^\infty$ sensitivity problem. It should be noted, however, that the techniques developed in Chapter 6 can be used to overcome these pathologies. Rather than taking this approach, we choose to use a nicer cost function; namely one which penalizes the control to reference transfer function rather than the complementary sensitivity transfer function. When this is done, the discontinuity exhibited in the above example does not occur. This has been shown in [53] under the uniform topology on $\mathcal{H}^\infty$. We shall show that useful results can be obtained when working with the compact topology on $\mathcal{H}^\infty$.

We now present our solution to the $\mathcal{H}^\infty$ *Approximate/Design Mixed-Sensitivity Problem.*

7.4 Solution to $\mathcal{H}^\infty$ Approximate/Design Mixed-Sensitivity Problem

In this section we shall solve th $\mathcal{H}^\infty$ *Approximate/Design Mixed-Sensitivity Problem.* We shall do so by constructing a near-optimal finite dimensional compensator for the infinite dimensional plant P. This will be done by appropriately modifying finite dimensional solutions $\{Q_n\}_{n=1}^\infty$ based on finite dimensional approximants $\{P_n\}_{n=1}^\infty$. The techniques developed in Chapter 5 shall be heavily exploited.

In this section, the following "compact convergence" assumption shall be made on the finite dimensional approximants $\{P_n\}_{n=1}^\infty$.

Assumption 7.4.1 (Construction of Approximants)
The sequence $\{P_n\}_{n=1}^\infty \in R\mathcal{H}^\infty$ is uniformly bounded in $\mathcal{H}^\infty$ and uniformly approximates P on all compact frequency intervals; i.e. for each $\Omega \in R_+$, however large, we have

$$\lim_{n\to\infty} \left\| (P_n - P)X_{[-\Omega,\Omega]} \right\|_{\mathcal{H}^\infty} = 0.$$

■

Comment 7.4.1 (Practicality)
Compact approximants are easily obtained. If P is a delay, for example, one can use Pade' approximants (cf. example 2.10.2). Other approximants can also be used (cf. example 2.10.1 and example 2.10.3).

■

The following theorem captures the main ideas in obtaining a solution to the $\mathcal{H}^\infty$ *Approximate/Design Mixed-Sensitivity Problem.* The theorem is analogous to theorem 6.4.1.

Theorem 7.4.1 (Main Ideas)
Suppose that

$$\lim_{n\to\infty} \mu_n \le \mu_{opt}, \tag{1}$$

and that there exists a uniformly bounded sequence $\{\tilde{Q}_n\}_{n=1}^\infty \subset R\mathcal{H}_0^\infty$ such that

$$\left\| \begin{pmatrix} W_1(1 - P_n\tilde{Q}_n) \\ W_2\tilde{Q}_n \end{pmatrix} \right\|_{\mathcal{H}^\infty} \le \mu_n + \epsilon, \tag{2}$$

for each n sufficiently large, and

$$\lim_{n\to\infty} \left\| (P_n - P)\tilde{Q}_n \right\|_{\mathcal{H}^\infty} = 0. \tag{3}$$

Then $\tilde{K}_n$ will "internally" stabilize P for all but a finite number of n and

$$\lim_{n\to\infty} \tilde{\mu}_n = \mu_{opt}.$$

Proof

The proof of this theorem is identical to the proof of theorem 6.4.1. ∎

Theorem 7.4.1 shows precisely what is needed to solve the $\mathcal{H}^\infty$ *Approximate/Design Mixed-Sensitivity Problem.* Comments analogous to those given for theorem 6.4.1 can be made here.

The following theorem shows how one might construct a strictly proper nearly-optimal compensator for the infinite dimensional plant P. We note that such a compensator is guaranteed to exist by the definition of the *optimal performance* μ_{opt} in definition 7.2.2. We present the theorem because it will provide insight into the subsequent construction of $r, \tilde{Q}_n$, and hence $\tilde{K}_n$.

Throughout the section, we shall require the following *roll-off* function.

Definition 7.4.1 (Roll-off Function)

$$h_m \stackrel{\text{def}}{=} \left(\frac{1}{s+1}\right)^{\frac{1}{m}}$$

where $m \in Z_+$.

∎

Such a function was discussed in Chapter 2 and heavily exploited in Chapters 5 and chapsensol.

Theorem 7.4.2 (Near-Optimal Irrational Compensator)

There exists $M_1 \stackrel{\text{def}}{=} M_1(\epsilon, \mu_{opt}) \in Z_+$ *such that*

$$\left\| \begin{pmatrix} W_1(1 - PQ_{opt}h_{M_1}) \\ W_2 Q_{opt} h_{M_1} \end{pmatrix} \right\|_{\mathcal{H}^\infty} \leq \mu_{opt} + 3\epsilon$$

If we define $\tilde{Q}_{opt} \stackrel{\text{def}}{=} Q_{opt} h_{M_1}$, *then* $\tilde{Q}_{opt} \in \mathcal{H}_0^\infty$ *and we have that*

$$\left\| \begin{pmatrix} W_1(1 - P\tilde{Q}_{opt}) \\ W_2 \tilde{Q}_{opt} \end{pmatrix} \right\|_{\mathcal{H}^\infty} \leq \mu_{opt} + 3\epsilon.$$

Given this,

$$\tilde{K}_{opt} \stackrel{\text{def}}{=} \frac{-\tilde{Q}_{opt}}{1 - P\tilde{Q}_{opt}}$$

will be a a strictly proper nearly-optimal compensator for P.

∎

Proof

Since the proof of this will be insightful, it shall be given. We have that $\tilde{Q}_{opt} \stackrel{\text{def}}{=} Q_{opt} h_m$ where $Q_{opt} \in \mathcal{H}^\infty$ achieves the infimum in definition 4.3.1. By definition $\tilde{Q}_{opt} \in \mathcal{H}_0^\infty$ and

$$\left\| \begin{pmatrix} W_1(1 - P\tilde{Q}_{opt}) \\ W_2 \tilde{Q}_{opt} \end{pmatrix} \right\|_{\mathcal{H}^\infty}^2 \stackrel{\text{def}}{=} \sup_{\omega \geq 0} |W_1(1 - PQ_{opt}h_m)|^2 + |W_2 Q_{opt} h_m|^2$$

To prove the result, it will suffice to show that

$$\left\| \begin{pmatrix} W_1(1 - PQ_{opt}h_m) \\ W_2 Q_{opt} h_m \end{pmatrix} \right\|_{\mathcal{H}^\infty}^2 \leq \mu_{opt}^2 + \epsilon$$

for m sufficiently large. To do this we first define

$$Y \stackrel{\text{def}}{=} W_1(1 - PQ_{opt}),$$

$$T_1 \stackrel{\text{def}}{=} W_1,$$

and

$$T \stackrel{\text{def}}{=} T_1(j\infty).$$

Given this, by adding and subtracting one gets the following

$$W_1(1 - PQ_{opt}h_m) = (T_1 - T)(1 - h_m) + T + (Y - T)h_m.$$

This then implies that

$$\left\| \begin{pmatrix} W_1(1 - PQ_{opt}h_m) \\ W_2 Q_{opt} h_m \end{pmatrix} \right\|_{\mathcal{H}^\infty}^2$$

$$\leq \sup_{\omega \geq 0} \{ |(T_1 - T)(1 - h_m)|^2 + 2|(T_1 - T)(1 - h_m)||T + (Y - T)h_m| + |T + (Y - T)h_m|^2 + |W_2 Q_{opt} h_m|^2 \}.$$

From this we obtain

$$\left\| \begin{pmatrix} W_1(1 - PQ_{opt}h_m) \\ W_2 Q_{opt} h_m \end{pmatrix} \right\|_{\mathcal{H}^\infty}^2 \leq \|(T_1 - T)(1 - h_m)\|_{\mathcal{H}^\infty}^2 + 2\|(T_1 - T)(1 - h_m)(T + (Y - T)h_m)\|_{\mathcal{H}^\infty}$$

$$+ \sup_{\omega \geq 0} \{ |T + (Y - T)h_m|^2 + |W_2 Q_{opt} h_m|^2 \}.$$

Since $T + (Y - T)h_m$ is uniformly bounded in $\mathcal{H}^\infty$ and since h_m approximates unity on compact frequency intervals, from lemma 2.10.1 it follows that

$$\left\| \begin{pmatrix} W_1(1 - PQ_{opt}h_m) \\ W_2 Q_{opt} h_m \end{pmatrix} \right\|_{\mathcal{H}^\infty}^2 \leq 2\epsilon + \sup_{\omega \geq 0} \{ \ |T + (Y - T)h_m|^2 + |W_2 Q_{opt} h_m|^2 \ \}$$

for m sufficently large. To prove the result we thus only need to consider the last two terms.

Using the algebraic result in lemma 5.2.2, we know that given $\epsilon > 0$, m can be chosen such that

$$|T + (Y - T)h_m|^2 \leq |T|^2(1 - 2|h_m| + |h_m|^2) + |Y|^2|h_m|^2 + 2|T||Y||h_m|(1 - |h_m|) + \epsilon$$

almost everywhere on the imaginary axis. This is the key step. It follows because

(1) $|T|$ and $|Y|$ are uniformly bounded,

(2) $\|h_m\|_{\mathcal{H}^\infty} \leq 1$, and

(3) h_m has special phase properties.

This, then implies that

$$|T + (Y - T)h_m|^2 + |W_2 Q_{opt}|^2 \leq |T|^2(1 - 2|h_m| + |h_m|^2) + 2|T||Y||h_m|(1 - |h_m|) + (|Y|^2 + |W_2 Q_{opt}|^2)|h_m|^2 + \epsilon$$

almost everywhere on the imaginary axis.

From proposition 2.7.2, it follows that

$$|T| \leq \mu_{opt}.$$

We also note that

$$|Y| \leq \mu_{opt}.$$

By definition, Q_{opt} satisfies

$$|Y|^2 + |W_2 Q_{opt}|^2 \leq \mu_{opt}^2 = \left\| \begin{pmatrix} W_1(1 - PQ_{opt}) \\ W_2 Q_{opt} \end{pmatrix} \right\|_{\mathcal{H}^\infty}^2$$

almost everywhere on the imaginary axis. Combining the above inequalities then gives

$$|T + (Y - T)h_m|^2 + |W_2 Q_{opt} h_m|^2 \leq \mu_{opt}^2(1 - 2|h_m| + |h_m|^2 + 2|h_m| - 2|h_m|^2) + \mu_{opt}^2 |h_m|^2 + \epsilon$$

almost everywhere on the imaginary axis. This then implies that

$$\sup_{\omega \geq 0} \{ \, |T + (Y - T)h_m|^2 + |W_2 Q_{opt} h_m|^2 \, \} \leq \mu_{opt}^2 + \epsilon$$

for m sufficently large. Given this, it then follows that m can be chosen such that

$$\left\| \begin{pmatrix} W_1(1 - PQ_{opt} h_m) \\ W_2 Q_{opt} h_m \end{pmatrix} \right\|_{\mathcal{H}^\infty}^2 \leq {\mu_{opt}}^2 + 3\epsilon.$$

This proves the result. ∎

Comment 7.4.2 (Main Ideas)
The main ideas in the above proof are as follows:
(1) μ_{opt} is finite,
(2) $\|h_m\|_{\mathcal{H}^\infty} \leq 1$,
(3) the phase properties of h_m.
These ideas shall be used shortly to get a result analogous to theorem 7.4.2.

∎

We now present the following "upper-semicontinuity" result for the mixed sensitivity problem. It gives us condition (1) in theorem 7.4.1.

Lemma 7.4.1 (Upper-Semicontinuity)

$$\lim_{n \to \infty} \mu_n \leq \mu_{opt}.$$

∎

Proof

From definition 7.2.2 for μ_{opt}, we know that given $\epsilon > 0$ there exists $Q_0 \in \mathcal{H}_0^\infty$ such that

$$\left\| \begin{pmatrix} W_1(1 - PQ_0) \\ W_2 Q_0 \end{pmatrix} \right\|_{\mathcal{H}^\infty} \leq \mu_{opt} + \epsilon.$$

We then have

$$\mu_n = \inf_{Q \in \mathcal{H}_0^\infty} \left\| \begin{pmatrix} W_1(1 - P_n Q) \\ W_2 Q \end{pmatrix} \right\|_{\mathcal{H}^\infty} \leq \left\| \begin{pmatrix} W_1(1 - P_n Q_0) \\ W_2 Q_0 \end{pmatrix} \right\|_{\mathcal{H}^\infty}$$

$$\leq \left\| \begin{pmatrix} W_1(1-PQ_0) \\ W_2Q_0 \end{pmatrix} \right\|_{\mathcal{H}^\infty} + \|W_1(P_n - P)Q_0\|_{\mathcal{H}^\infty}$$

which after using the near optimality of Q_0 gives

$$\leq \mu_0 + \epsilon + \|W_1(P_n - P)Q_0\|_{\mathcal{H}^\infty}$$

Finally, by assumption 7.4.1 on the approximants, the form of $\tilde{Q}_n$, and lemma 2.10.1 we can make the right term arbitrarily small by taking n sufficiently large. Since ϵ can be arbitrarily small this proves the result. ∎

Comment 7.4.3 (Uniform Boundedness)
Lemma 7.4.1 implies that μ_n is a uniformly bounded sequence. This fact is crucial in the proof of the following lemma.

∎

The following lemma is analogous to lemma ?? and gives us the "sub-optimality" condition (2) in theorem 7.4.1.

Lemma 7.4.2 (Near Sub-Optimality)
There exists $f \in R\mathcal{H}_0^\infty$ such that

$$\left\| \begin{pmatrix} W_1(1-P_nQ_nf) \\ W_2Q_nf \end{pmatrix} \right\|_{\mathcal{H}^\infty} \leq \mu_n + \epsilon,$$

for n sufficiently large. Moreover, if we define a roll-off operator $r : Q_n \to \tilde{Q}_n \in R\mathcal{H}_0^\infty$ where

$$\tilde{Q}_n \stackrel{\text{def}}{=} Q_nf,$$

then

$$\left\| \begin{pmatrix} W_1(1-P_n\tilde{Q}_n) \\ W_2\tilde{Q}_n \end{pmatrix} \right\|_{\mathcal{H}^\infty} \leq \mu_n + \epsilon$$

for each $n \in Z_+$. Given this, $\tilde{K}_n$ will be a finite dimensional strictly proper near-optimal compensator for P_n for each $n \in Z_+$.

∎

Proof The proof is identical to that given for theorem 7.4.2. The "upper-semicontinuity" result of 7.4.1 guarantees that μ_n will be bounded. This is critical in the proof. See comment 7.4.2. Here one can use $f \stackrel{\text{def}}{=} \tilde{h}_m$ where $\tilde{h}_m$ is simply an $R\mathcal{H}^\infty$ function which is sufficiently "close" to the irrational function h_m in the $\mathcal{H}^\infty$ topology. Such a function is guaranteed to exist by lemma 2.10.1. ∎

Given the above, we can obtain the "stability" condition (3) in theorem 7.4.1.

Lemma 7.4.3 (Internal Stability)

The sequence $\{\tilde{Q}_n\}_{n=1}^\infty \subset R\mathcal{H}_0^\infty$ is uniformly bounded and uniformly rolls-off in $\mathcal{H}^\infty$. Moreover,

$$\lim_{n\to\infty} \left\| (P_n - P)\tilde{Q}_n \right\|_{\mathcal{H}^\infty} = 0.$$

■

Proof

Since W_2 is invertible in $\mathcal{H}^\infty$ we have the following:

$$\|Q_n\|_{\mathcal{H}^\infty} \le \left\|W_2^{-1}\right\|_{\mathcal{H}^\infty} \|W_2 Q_n\|_{\mathcal{H}^\infty} \le \left\|W_2^{-1}\right\|_{\mathcal{H}^\infty} \left\| \begin{pmatrix} W_1(1 - P_n Q_n) \\ W_2 Q_n \end{pmatrix} \right\|_{\mathcal{H}^\infty} \le \left\|W_2^{-1}\right\|_{\mathcal{H}^\infty} \mu_n.$$

Since $\lim_{n\to\infty} \mu_n \le \mu_{opt}$ (cf. lemma 7.4.1), we can conclude that Q_n is uniformly bounded in $\mathcal{H}^\infty$. The result then follows from the form of $\tilde{Q}_n$, assumption 7.4.1 on the approximants, and lemma 2.10.1 which allows us to turn compact convergence into uniform convergence. ■

We now state the main result of this chapter which guarantees the existence of a near-optimal strictly proper finite dimensional compensator.

Theorem 7.4.3 (Main Result: Solution to Approximate/Design Problem)
There exists a roll-off operator $r : Q_n \to \tilde{Q}_n \in R\mathcal{H}_0^\infty$ such that

$$\lim_{n\to\infty} \tilde{\mu}_n = \mu_{opt}.$$

Moreover, the $\tilde{Q}_n$ constructed in lemma 7.4.2 define such an operator.

■

Proof

The proof follows from theorem 7.4.1 which captures the main ideas, the results in lemmas 7.4.1 , 7.4.2, and 7.4.3 and the construction of the sequence $\{\tilde{Q}_n\}_{n=1}^{\infty}$. ■

This solves the $\mathcal{H}^\infty$ *Approximate/Design Mixed-Sensitivity Problem.*

7.5 Solution to $\mathcal{H}^\infty$ Purely Finite Dimensional Mixed-Sensitivity Problem: Computation of Optimal Performace

In practice we often would like to compute or estimate the *optimal performance*, μ_{opt}. The following theorem says that the *expected performance*, μ_n actually approaches the *optimal performance*, μ_{opt}.

Theorem 7.5.1 (Solution to Purely Finite Dimensional Problem)

$$\lim_{n\to\infty} \mu_n = \mu_{opt}.$$

■

Proof

We have already shown in lemma 7.4.1 that

$$\lim_{n\to\infty} \mu_n \le \mu_{opt}.$$

The proof of the converse inequality parallels the proof given in theorem 6.5.1. ■

This solves the $\mathcal{H}^\infty$ *Purely Finite Dimensional Mixed-Sensitivity Problem.*

Usually the *optimal performance* μ_{opt} is computed by solving an infinite dimensional eigenvalue/eigenfunction problem involving a Hankel - Toeplitz operator pair [40]-[41], [61]. The above theorem shows that μ_{opt} can be computed by solving a sequence of finite dimensional problems. As in corollary 6.5.1, one can associate finite dimensional eigenvalue/eigenvector problems with these finite dimensional problems. Consequently, to estimate μ_{opt}, we only need to solve a sequence of finite dimensional eigenvalue/eigenvector problems. Moreover, the ideas of [12] can be used to assist with numerical computations.

7.6 Poles and Zeros on the Imaginary Axis

Unlike with the $\mathcal{H}^\infty$ sensitivity problem, the mixed-sensitivity problem does not suffer from imaginary axis pole-zero discontinuities.

7.7 Solution to $\mathcal{H}^\infty$ Loop Convergence Mixed-Sensitivity Problem

Thus far, we have given conditions under which the *expected performace* μ_n and the *actual performance* $\tilde{\mu}_n$ approach the *optimal performance* μ_{opt}. Results analagous to those in section 6.6 can be obtained which guarantee the convergence of the actual loop shapes. Moreover, it is not necessary that P_n approximate P uniformly.

7.8 Unstable Plants

Unstable plants can be treated similarly. Additional care must be taken to guarantee stability. This is done by approximating the factors N_k, D_k, D_p uniformly. N_p need only be approximated on compact frequency intervals.

7.9 General Weighting Functions

The theory presented in this and the previous chapter can be generalized to handle more general weights. Weights in $R\mathcal{H}^\infty$ offer a designer a considerable amount of flexibility. Such an extension will not be pursued in this work.

7.10 Super-Optimal Performance Criteria

Often, we might like to optimize a given performance criterion subject to an additional constraint. For example, $\mathcal{H}^\infty$ sensitivity minimization subject to an $\mathcal{H}^\infty$ or an $\mathcal{L}^1$ bound. Given the ability to obtain proper finite dimensional compensators for finite dimensional versions of such problems, the techniques presented thus far can be used to construct strictly proper finite dimensional near-optimal compensators for the infinite dimensional plant. Such performance criteria are well-posed in the sense of [53]; i.e. they are continuous with respect to plant perturbations in the uniform topology on $\mathcal{H}^\infty$.

7.11 Summary

In this chapter solutions were presented to the $\mathcal{H}^\infty$ *Approximate/Design* , *Purely Finite Dimensional, and Loop Convergence Mixed-Sensitivity Problems.*

Chapter 8

Design via $\mathcal{H}^2$ Optimization

8.1 Introduction

In this chapter we consider the problem of designing near-optimal finite dimensional compensators for infinite dimensional plants via $\mathcal{H}^2$ optimization. Such an approach can be motivated by design specifications which require some specified degree of robustness or $\mathcal{L}^2$ disturbance rejection. A systematic procedure is presented. More specifcally, we provide a solution to the $\mathcal{H}^2$ *Approximate/Design Sensitivity Problem*, the $\mathcal{H}^2$ *Purely Finite Dimensional Sensitivity Problem*, and the $\mathcal{H}^2$ *Loop Convergence Sensitivity Problem.* Solutions to the corresponding Mixed-sensitivity problems shall also be disussed. Again, we focus on stable plants in order to isolate the key ideas.

8.2 $\mathcal{H}^2$ Approximate/Design Sensitivity Problem

In this section we present some definitions and assumptions to precisely state the $\mathcal{H}^2$ *Approximate/Design Sensitivity Problem.* Notation to be used throughout the chapter is also established.

Since we shall assume that our infinite dimensional plant is stable we have $P \in \mathcal{H}^\infty$. Given this, it follows from proposition 3.2.1 that the set of all compensators which internally stabilize P, with respect to the ring $\mathcal{H}^\infty$, are parameterized by

$$K(P,Q) \stackrel{\text{def}}{=} \frac{-Q}{1-PQ}$$

where Q is any element in $\mathcal{H}^\infty$. From this, it follows that if we allow Q to vary over $\mathcal{H}_0^\infty$, then we get all strictly proper compensators which internally stabilize P. We shall be doing this throughout the chapter; i.e. all infimizations involving P shall be carried out over $\mathcal{H}_0^\infty$.

In this chapter we shall construct $R\mathcal{H}^\infty$ approximants $\{P_n\}_{n=1}^\infty$ for P. From proposition 3.2.1, it follows that the set of all compensators which internally stabilize P_n, with respect to the ring $\mathcal{H}^\infty$, are parameterized by

$$K(P_n,Q) \stackrel{\text{def}}{=} \frac{-Q}{1-P_nQ}$$

where Q is any element in $\mathcal{H}^\infty$. From this, it follows that if we allow Q to vary over $\mathcal{H}_0^\infty$, then we get all strictly proper compensators which internally stabilize P_n. We shall be doing this throughout the chapter; i.e. all infimizations involving P_n shall be carried out over $\mathcal{H}_0^\infty$.

In this section we shall formulate an $\mathcal{H}^2$ weighted sensitivity problem. To do so, we shall require a frequency dependent *weighting function* W. The following assumption shall be made on W.

Assumption 8.2.1 (Weighting Function)

(1) $W \in R\mathcal{H}^2$.
(2) W is outer.
(3) $W(s) = \sum_{k=1}^{l} \frac{c_k}{s+a_k}$ where $c_k \in C$ and $Re(a_k) \leq 0$.

∎

We now define the notion of an $\mathcal{H}^2$*-sensitivity measure* as follows.

Definition 8.2.1 ($\mathcal{H}^\infty$-Sensitivity Measure)
Let $Q \in \mathcal{H}^\infty$ and $F, G, \in \mathcal{H}^\infty$. Also, let $K(G,Q)$ denote a compensator which internally stabilizes G with respect to the ring $\mathcal{H}^\infty$. If it also internally stabilizes F, it is appropriate to define the $\mathcal{H}^2$-sensitivity measure of the pair (F, K(G,Q)) as follows:

$$J_{\mathcal{H}^2}(F, K(G,Q)) \stackrel{\text{def}}{=} \left\| \frac{W}{1 - FK(G,Q)} \right\|_{\mathcal{H}^2}.$$

∎

From definition 8.2.1 above, it follows that $J_{\mathcal{H}^2}(P, K(P,Q)) \stackrel{\text{def}}{=} \left\| \frac{W}{1-PK(P,Q)} \right\|_{\mathcal{H}^2}$. Substituting into definition 4.3.1 and allowing Q to vary over $\mathcal{H}_0^\infty$, then gives us the following expression for the *optimal performance*, μ_{opt}.

Definition 8.2.2 (Optimal Performance)

$$\mu_{opt} \stackrel{\text{def}}{=} \inf_{Q \in \mathcal{H}_0^\infty} \left\| \frac{W}{1 - PK(P,Q)} \right\|_{\mathcal{H}^2} = \inf_{Q \in \mathcal{H}_0^\infty} \|W(1 - PQ)\|_{\mathcal{H}^2}.$$

∎

We emphasize that this definition defines an infinite dimensional optimization problem; one which we want to, and will, avoid solving. Moreover, we note that the problem is an $\mathcal{H}^2$ *Model Matching Problem.* It is analogous to the $\mathcal{H}^\infty$ problem studied in section 5.2.

Similarly, from definition 8.2.1, it follows that $J_{\mathcal{H}^2}(P_n, K(P_n,Q)) \stackrel{\text{def}}{=} \left\| \frac{W}{1-P_nK(P_n,Q)} \right\|_{\mathcal{H}^2}$. After substituting this into definition 5.5.1, and allowing Q to vary over $R\mathcal{H}_0^\infty$, one obtains the following expression for the *expected performance*, μ_n.

Definition 8.2.3 (Expected Performance)

$$\mu_n \stackrel{\text{def}}{=} \inf_{Q \in R\mathcal{H}_0^\infty} \left\| \frac{W}{1 - P_nK(P_n,Q)} \right\|_{\mathcal{H}^2} = \inf_{Q \in R\mathcal{H}_0^\infty} \|W(1 - P_nQ)\|_{\mathcal{H}^2}.$$

∎

Here, we can infimize over $R\mathcal{H}_0^\infty$ since W and P_n are real-rational. We note that this definition defines a sequence of finite dimensional model matching problems.

An optimal or near-optimal solution to this problem is typically found by first considering the "inner problem":

$$\inf_{Z \in R\mathcal{H}^\infty} \|W - P_{n_i}Z\|_{\mathcal{H}^2}.$$

Here, P_{n_i} is the inner part of P_n.

From the *classical projection theorem*, we know that there exists $Z_n \in R\mathcal{H}_0^\infty$ such that

$$\|W - P_{n_i} Z_n\|_{\mathcal{H}^2} = \min_{Z \in R\mathcal{H}^\infty} \|W - P_{n_i} Z\|_{\mathcal{H}^2} .$$

Moreover, Z_n is unique and is given by

$$Z_n = \Pi_{\mathcal{H}^2} W P_{n_i}^* .$$

Lets define

$$Q_n \stackrel{\text{def}}{=} W^{-1} P_{n_o}^{-1} Z_n$$

where P_{n_o} is the outer part of P_n. This Q_n generates a finite dimensional compensator

$$K_n \stackrel{\text{def}}{=} K(P_n, Q_n) = \frac{-Q_n}{1 - P_n Q_n}.$$

This compensator, as we expected, need not even stabilize P. We thus need to modify it. For this reason, we define a *roll-off operator*

$$r : Q_n \to \tilde{Q}_n \in R\mathcal{H}_0^\infty .$$

The exact form of r will be determined subsequently.

The compensator generated by $\tilde{Q}_n$ is given by

$$\tilde{K}_n \stackrel{\text{def}}{=} K(P_n, \tilde{Q}_n) = \frac{-\tilde{Q}_n}{1 - P_n \tilde{Q}_n}.$$

Given this, we consider the feedback system obtained by substituting $\tilde{K}_n$ into a closed loop system with the infinite dimensional plant P. Let $H(P, \tilde{K}_n)$ denote the resulting closed loop transfer function matrix from r, d to e, u. We then have

$$\begin{bmatrix} e \\ u \end{bmatrix} = H(P, \tilde{K}_n) \begin{bmatrix} r \\ d \end{bmatrix},$$

where

$$H(P, \tilde{K}_n) = \begin{bmatrix} \frac{1}{1-P\tilde{K}_n} & \frac{P}{1-P\tilde{K}_n} \\ \frac{\tilde{K}_n}{1-P\tilde{K}_n} & \frac{1}{1-P\tilde{K}_n} \end{bmatrix}.$$

Substituting for $\tilde{K}_n$, then gives

$$H(P, \tilde{K}_n(\tilde{Q}_n)) = \begin{bmatrix} 1 - P_n\tilde{Q}_n & P(1 - P_n\tilde{Q}_n) \\ -\tilde{Q}_n & 1 - P_n\tilde{Q}_n \end{bmatrix} \frac{1}{1 - (P_n - P)\tilde{Q}_n}.$$

Given that internal stability can be shown, the *actual performance*, $\tilde{\mu}_n$ defined in definition 4.3.3 is well defined and becomes:

Definition 8.2.4 (Actual Performance)

$$\tilde{\mu}_n \stackrel{\text{def}}{=} \left\| \frac{W}{1 - P\tilde{K}_n} \right\|_{\mathcal{H}^2} = \left\| \frac{W(1 - P_n\tilde{Q}_n)}{1 - (P_n - P)\tilde{Q}_n} \right\|_{\mathcal{H}^2}$$

■

Given the above definitions, the $\mathcal{H}^2$ *Approximate/Design Sensitivity Problem* then becomes to find conditions on the approximants $\{P_n\}_{n=1}^{\infty}$, and on the *roll-off operator* r, such that the *actual performance* approaches the *optimal performance*; i.e.

$$\lim_{n\to\infty} \tilde{\mu}_n = \mu_{opt}.$$

Equivalently, this problem can be viewed as that of finding a near-optimal compensator for the infinite dimensional plant P. The problem also addresses the question: What is a "good" finite dimensional approximant?

Because this problem is of primary concern in this research, we now indicate what difficulties are associated with the problem.

8.3 Why is the Approximate/Design Problem Hard?

There are several reasons one can give to illustrate the difficulties associated with the *Approximate/Design Problem.* We now discuss some of these.

First, one must note that the weighted $\mathcal{H}^2$ sensitivity problem, in general, is discontinuous with respect to plant perturbations, even when the uniform topology on $\mathcal{H}^\infty$ is imposed. This has been demonstrated in [53]. Consequently, simple continuity arguments cannot be used to obtain a solution to our *Approximate/Design Problem.* It must also be noted, however, that even if it were continuous in the uniform topology, there are many infinite dimensional plants which cannot be approximated uniformly by real-rational functions (e.g. a delay; see proposition 2.10.1).

Another difficulty can be attributed to the fact that weighted $\mathcal{H}^2$ optimal solutions generally exhibit bad properties. More specifically, one can show that the optimal solution is often unbounded and results in an improper compensator. One can correctly argue that this is usually an existence issue, nevertheless, it is an issue which a designer must contend with.

The following example illustrates that even uniform approximations can lead to bad results [53].

Example 8.3.1 (Discontinuity of $\mathcal{H}^2$ Sensitivity Problem)

Let our infinite dimensional plant be given by $P(s) = \frac{e^{-s}}{s+1}$. Let the weighting function be given by $W = \frac{1}{s+1}$. The associated optimal compensator can be found by solving the infinite dimensional model matching problem defined by:

$$\mu_{opt} \stackrel{\text{def}}{=} \inf_{Q\in\mathcal{H}_0^\infty} \|W(1-PQ)\|_{\mathcal{H}^2}.$$

Using classical projection theory, one obtains

$$\left\|\frac{1}{s+1}(1-\frac{e^{-s}}{s+1}Q)\right\|_{\mathcal{H}^2}^2 = \left\|\frac{e^s}{s+1} - \frac{1}{(s+1)^2}Q\right\|_{\mathcal{H}^2}^2 = \left\|\frac{e^s}{s+1} - \frac{e^{-1}}{s+1}\right\|_{\mathcal{H}^2}^2 + \left\|\frac{e^{-1}}{s+1} - \frac{1}{(s+1)^2}Q\right\|_{\mathcal{H}^2}^2.$$

$$\geq (1+e^{-2})\pi + \left\|\frac{e^{-1}}{s+1} - \frac{1}{(s+1)^2}Q\right\|_{\mathcal{H}^2}^2.$$

The last term can be made arbitrarily small by appropriate choice of Q. Consequently,

$$\mu_{opt} = \left\|\Pi_{\mathcal{H}^{2\perp}} W e^s\right\|_{\mathcal{H}^2} = (1+e^{-2})\pi.$$

We want to obtain a near-optimal finite dimensional compensator, by solving an appropriately formulated finite dimensional problem. Let $P_n(s) = (\frac{n}{s+n})^n \frac{1}{s+1}$ define a set of finite dimensional approximants for P. It can be shown that P_n uniformly aprroximates P on the extended imaginary axis (cf. example 2.10.1). The approximants P_n are thus terrific, based on "open-loop intuition". However, simple analysis shows that $\mu_n = \left\|\Pi_{\mathcal{H}^2 \perp} W P_{n_i}^*\right\|_{\mathcal{H}^2} = 0 < \mu_{opt}$ for each $n \in Z_+$. We thus have a discontinuity; i.e. μ_n does not approach μ_{opt} for large n, even though P_n approaches P.

Although the approximants selected appeal to our open loop intuition, it is clear that they do not help in achieving the closed loop objective. The approximants selected are bad because they fail to approximate the inner part of the plant. Doing so is important when solving an $\mathcal{H}^2$ sensitivity problem.

■

The above example clearly shows that the approximants P_n must be chosen appropriately. Approximants must be chosen on the basis of a closed loop design objective; not on the basis of open loop intuition. Consequently, which approximants are used is critically dependent on which design criterion is used.

Finally, the example shows that simple continuity arguments can not be used to obtain a solution to our Approximate/Design problem.

We now present our solution to the $\mathcal{H}^2$ *Approximate/Design Sensitivity Problem.*

8.4 Solution to $\mathcal{H}^2$ Approximate/Design Sensitivity Problem

In this section we shall solve the $\mathcal{H}^2$ *Approximate/Design Sensitivity Problem.* We shall do so by constructing a near-optimal finite dimensional compensator for the infinite dimensional plant P. This will be done by appropriately modifying finite dimensional solutions $\{Q_n\}_{n=1}^{\infty}$ based on the finite dimensional approximants $\{P_n\}_{n=1}^{\infty}$. The techniques developed in Chapter 5 shall be heavily exploited.

In this section, the following assumption will be made about the infinite dimensional plant P and the finite dimensional approximants $\{P_n\}_{n=1}^{\infty}$.

Assumption 8.4.1 (Construction of Approximants and Bezout Factors)

(1) P has a finite number of zeros on the extended imaginary axis; each with finite algebraic multiplicity.

(2) The sequence $\{P_{n_i}\}_{n=1}^{\infty} \subset R\mathcal{H}^{\infty}$ consists of inner functions which uniformly approximate P_i on all compact frequency intervals (excluding the point $j\infty$); i.e. for each $\Omega \in R_+$, however large, we have $\lim_{n\to\infty} \left\|(P_{n_i} - P_i)X_{[-\Omega,\Omega]}\right\|_{\mathcal{H}^{\infty}} = 0$.

(3) The sequence $\{P_{n_o}\}_{n=1}^{\infty} \subset R\mathcal{H}^{\infty}$ consists of outer functions which uniformly approximate P_o; i.e. $\lim_{n\to\infty} \|P_{n_o} - P_o\|_{\mathcal{H}^{\infty}} = 0$. Moreover, the sequence is constructed as indicated in construction 5.5.1.

■

Comment 8.4.1 (Applicability, Practicality)

(a) Zeros on Extended Imaginary Axis.

Relaxing condition (1) will be an area for future research.

(b) Inner Part and Right Half Plane Zeros.

Condition (2) is reasonable since it allows N_{p_i} to be discontinuous at ∞. It thus allows for plants with delays. Delays can be approximated uniformly on compact frequency intervals using Pade' approximants (cf. example 2.10.2). Such approximants agree with control engineering intuition: the need to approximate the plant at "low" frequencies.

The condition allows P to have an infinite number of zeros in the open right half plane. P , for example, may contain an infinite Blaschke product of open right half plane zeros provided that the zeros accumulate at ∞ only. In such a case, the partial products can be used as the $N_{p_{n_i}}$ [30].

If P has an infinite number of open right half plane zeros, then the zeros can only accumulate on the imaginary axis or at ∞. If they were to accumulate within the finite open right half plane, then this would imply that N_{p_i} is identically zero in the open right half plane (cf. proposition 2.3.5). If they were to accumulate on the imaginary axis then P_i would possess essential singularities at those points and hence we could not approximate it uniformly on compact frequency intervals (cf. proposition 2.7.3). It thus follows that the only point of accumulation can be ∞.

(c) Continuity of P_o.

It should be noted that the approximants in (3) are guaranteed to exist if and only if $P_o \in \mathcal{C}_e$. This follows from proposition 2.10.1.

(d) Approximation of Inner and Outer Parts.

We approximate the inner and outer parts separately, in order to control the pole-zero structure of the approximants P_n on the imaginary axis. If we did not perform the approximations in the above manner, then the pole-zero structure of P_n and P may differ drastically on the imaginary axis, even for large n. Such a situation is highly undesirable since it would complicate the inversion of P_{n_o}. We would like to invert P_{n_o} as we invert P_o. To do so, they must possess similar imaginary zero structures. This was seen in Chapter 6. Since the approximants are based on the construction given in construction 5.5.1, the ideas in construction 5.5.1 are critical.

(e) Compact Approximants.

Finally , we note that (2) and (3) imply that the sequence $\{P_n\}_{n=1}^{\infty} \subset R\mathcal{H}^\infty$ uniformly approximates P on all compact frequency intervals (excluding the point $j\infty$); i.e. for each $\Omega \in R_+$, however large, we have $\lim_{n\to\infty} \left\|(P_n - P)X_{[-\Omega,\Omega]}\right\|_{\mathcal{H}^\infty} = 0$.

∎

From proposition 3.2.1, it follows that the set of all compensators which internally stabilize P_n,

with respect to the ring $R\mathcal{H}^\infty$, are parameterized by

$$K(P_n, Q) \stackrel{\text{def}}{=} \frac{-Q}{1 - P_n Q}$$

where Q is any element in $R\mathcal{H}^\infty$ [56]. Consequently, the approximants P_n, as constructed above, possess the desired algebraic properties.

The following theorem captures the main ideas in obtaining a solution to the $\mathcal{H}^2$ *Approximate/Design Sensitivity Problem.*

Theorem 8.4.1 (Main Ideas)
Suppose that

$$\lim_{n\to\infty} \mu_n \leq \mu_{opt} \tag{1}$$

and that there exists a uniformly bounded sequence $\{\tilde{Q}_n\}_{n=1}^{\infty} \subset R\mathcal{H}_0^\infty$ *such that*

$$\left\| W(1 - P_n \tilde{Q}_n) \right\|_{\mathcal{H}^2} \leq \mu_n + \epsilon, \tag{2}$$

for n sufficiently large, and

$$\lim_{n\to\infty} \left\| (P_n - P)\tilde{Q}_n \right\|_{\mathcal{H}^\infty} = 0. \tag{3}$$

Given the above, $\{\tilde{K}_n\}_{n=1}^{\infty}$ *will internally stabilize P with respect to the ring $\mathcal{H}^\infty$ for all but a finite number of n. In addition, the actual performance approaches the optimal performance as the approximants get "better"; i.e.*

$$\lim_{n\to\infty} \tilde{\mu}_n = \mu_{opt}.$$

■

Proof

Since

$$\mu_{opt} \stackrel{\text{def}}{=} \inf_{Q\in\mathcal{H}_0^\infty} \left\| \frac{W}{1 - PK(P,Q)} \right\|_{\mathcal{H}^2}$$

and

$$\tilde{\mu}_n \stackrel{\text{def}}{=} \left\| \frac{W}{1 - PK(P_n, \tilde{Q}_n)} \right\|_{\mathcal{H}^2} = \left\| \frac{W(1 - P_n\tilde{Q}_n)}{1 - (P_n - P)\tilde{Q}_n} \right\|_{\mathcal{H}^2}$$

we have $\mu_{opt} \leq \tilde{\mu}_n$ for each $n \in Z_+$. Consequently,

$$\mu_{opt} \leq \tilde{\mu}_n \leq \frac{\left\| W(1 - P_n\tilde{Q}_n) \right\|_{\mathcal{H}^2}}{1 - \left\| (P_n - P)\tilde{Q}_n \right\|_{\mathcal{H}^\infty}}.$$

The result then follows from conditions (1), (2), and (3) within the theorem.

We now show that (3) implies internal stability. To do so, we argue as follows. Since $\tilde{K}_n$ internally stabilizes P_n, it follows that $-\tilde{Q}_n$ and $1 - P_n\tilde{Q}_n$ must be coprime in $\mathcal{H}^\infty$. We also have that P_n and 1 are coprime in $\mathcal{H}^\infty$. Consequently, from proposition 3.3.1, we have that $\tilde{K}_n$ internally stabilizes P if and only if $1 - (P_n - P)\tilde{Q}_n$ is a unit of (i.e. invertible in) $\mathcal{H}^\infty$. However, (3) implies that $1 - (P_n - P)\tilde{Q}_n$ will be a unit for all but a finite number of n. This completes the proof. ■

Theorem 8.4.1 shows precisely what is needed to solve the $\mathcal{H}^2$ *Approximate/Design Sensitivity Problem.* It was shown in section 6.3 that one can run into serious difficulty satisfying the third

condition of theorem 8.4.1 if one chooses $\tilde{Q}_n = Q_n$. We emphasize that it is condition (3) which will allow us to guarantee internal stability when $\tilde{K}_n$ is used with P. Because of this, the *"traditional" choice of r, as the identity, comes with no guarantees.* In what follows we shall present a way for choosing $\tilde{Q}_n$, and hence the roll-off operator r, so that all three conditions in theorem 8.4.1 are satisfied. With this construction we will have a solution to the $\mathcal{H}^2$ *Approximate/Design Sensitivity Problem.*

We shall see that condition (1) in theorem 8.4.1 will follow from the implicit structure of the weighted sensitivity problem. This condition should be interpreted loosely as an "upper-semicontinuity" condition. Such a condition should be expected from the results in [53, pp. 345].

Condition (2) in theorem 8.4.1 will be achieved by exploiting the ideas developed in Chapter 5. We refer to condition (2) as a "sub-optimality" condition.

Finally, condition (3) in theorem 8.4.1 will follow after showing that the $\tilde{Q}_n$ are uniformly bounded and uniformly roll-off in $R\mathcal{H}_0^\infty$. This will be critical in stablishing (3). Since condition (3) gives us stability, we shall refer to it as the "internal stability" condition.

The main result of this section is now stated in the following theorem.

Theorem 8.4.2 (Main Result: Solution to Approximate/Design Problem)

There exists a sequence $\{\tilde{Q}_n\}_{n=1}^{\infty} \subset R\mathcal{H}_0^\infty$ which is uniformly bounded, uniformly rolls-off. There also exists $N \in Z_+$ such that

$$\mu_{opt} \leq \tilde{\mu}_n \leq \mu_{opt} + \epsilon$$

for all $n \geq N$. Consequently, there exists a roll-off operator $r : Q_n \to \tilde{Q}_n \in R\mathcal{H}_0^\infty$ such that

$$\lim_{n\to\infty} \tilde{\mu}_n = \mu_{opt}.$$

Moreover, the sequence $\{\tilde{Q}_n\}_{n=1}^{\infty}$ generates a sequence of finite dimensional, strictly proper, internally stabilizing compensators $\{\tilde{K}_n\}_{n=1}^{\infty}$, where

$$\tilde{K}_n = \frac{-\tilde{Q}_n}{1 - P_n\tilde{Q}_n}$$

and

$$\mu_{opt} \leq \left\| \frac{W}{1 - P\tilde{K}_n} \right\|_{\mathcal{H}^2} \leq \mu_{opt} + \epsilon$$

for all $n \geq N$. $\tilde{K}_n$ is thus nearly-optimal for all $n \geq N$.

■

Proof

To prove the theorem, we only need to show that conditions (1), (2), and (3) of theorem 8.4.1 can be satisfied. We proceed in six steps.

Step 1: Upper-semicontinuity Condition.

Let $Q_o \in \mathcal{H}_o^\infty$ be such that $\mu_{opt} \leq \|W(1 - PQ_o)\|_{\mathcal{H}^2} + \epsilon$. The upper-semicontinuity condition then follows from the following inequality

$$\mu_n \leq \|W(1 - P_nQ_o)\|_{\mathcal{H}^2} \leq \|W(1 - PQ_o)\|_{\mathcal{H}^2} + \|W(P_n - P)Q_o)\|_{\mathcal{H}^2}.$$

Step 2: Uniform Boundedness and Uniform Roll-off.

Given that $Z_n = \Pi_{\mathcal{H}^2} W P_{n_i}^*$, we note that

$$Z_n = \sum_{k=1}^{l} c_k \frac{P_{n_i}^*(-a_k)}{s + a_k}.$$

It thus follows that Z_n is uniformly bounded and uniformly rolls-off in $\mathcal{H}^\infty$.

Let $\tilde{Q}_n \stackrel{\text{def}}{=} Z_n W^{-1} P_{n_o}^{-1} g$ where $g \in R\mathcal{H}^\infty$. We know from Chapter 5 that g can be chosen such that $W^{-1} P_{n_o}^{-1} g$ is uniformly bounded in $\mathcal{H}^\infty$. Thus $\tilde{Q}_n$ is uniformly bounded and uniformly rolls-off in $\mathcal{H}^\infty$.

Step 3: Computation of μ_n.

Simple analysis shows that $\mu_n = \left\| \Pi_{\mathcal{H}^{2\perp}} W P_{n_i}^* \right\|_{\mathcal{H}^2} = \| W - P_{n_i} Z_n \|_{\mathcal{H}^2}$.

Step 4: Sub-optimality Condition.

We note that

$$\left\| W(1 - P_n \tilde{Q}_n) \right\|_{\mathcal{H}^2} \leq \| W - P_{n_i} Z_n \|_{\mathcal{H}^2} + \| Z_n (1 - g) \|_{\mathcal{H}^2}.$$

The first term is μ_n. The second term can be made arbitrarily small by choosing g appropriately. That such a function g exists follows from Chapter 5. We thus have

$$\left\| W(1 - P_n \tilde{Q}_n) \right\|_{\mathcal{H}^2} \leq \mu_n + \epsilon$$

for all $n \in Z_+$. This gives the sub-optimality condition (2).

Step 5: Internal Stability Condition.

To complete the proof, we only need to show that condition (3) of theorem 8.4.1 holds; i.e.

$$\lim_{n \to \infty} \left\| (P_n - P) \tilde{Q}_n \right\|_{\mathcal{H}^\infty} = 0.$$

This, however, follows since $\tilde{Q}_n$ uniformly rolls-off and since P_n approximates P uniformly on compact frequency intervals (cf. lemma 2.10.2).

Step 6: Conclusion.

Given the above, it follows from theorem 8.4.1 that $\{\tilde{K}_n\}_{n=1}^{\infty}$ will internally stabilize P with respect to the ring $\mathcal{H}^\infty$ for all but a finite number of n. In addition, the *actual performance* approaches the *optimal performance* as the approximants get "better"; i.e.

$$\lim_{n \to \infty} \tilde{\mu}_n = \mu_{opt}.$$

This completes the proof. ∎

The above theorem shows that given our assumptions on the weighting function, the approximants, and on the plant, there exists a way to construct nearly-optimal, finite dimensional, strictly proper controllers for the infinite dimensional plant; i.e. in a manner such that the *actual performance* approaches the *optimal performance* as the approximants get better. Consequently, we have solved the $\mathcal{H}^2$ *Approximate/Design Sensitivity Problem.* It should be emphasized that this has been done by using approximants which converge in the compact topology on $\mathcal{H}^\infty$ and do not necessarily converge in the uniform topology. Moreover, the conditions which the approximants $\{P_n\}_{n=1}^{\infty}$ must satisfy are weak.

Comment 8.4.2 (Direct Construction of Finite Dimensional Compensators)
From theorem 5.2.2, it immediately follows that we can also construct nearly-optimal infinite dimensional compensators for P. In principle, one can approximate such compensators to get finite dimensional compensators. This, however, would defeat our purpose. In our *Approximate /Design* approach we intentionally avoid solving infinite dimensional optimization problems.

To perform the above construction would mean that we would have to solve an infinite dimensional "inner problem" for Z_{opt} (cf. proposition 5.2.1). In the context of this work, however, this is unacceptable.

∎

8.5 Solution to $\mathcal{H}^2$ Purely Finite Dimensional Sensitivity Problem: Computation of Optimal Performance

In practice we often would like to compute or estimate the *optimal performance* μ_{opt}. The following theorem says that under our assumptions, the *expected performance* μ_n approaches the *optimal performance* μ_{opt}.

Theorem 8.5.1 (Solution to Purely Finite Dimensional Problem)

$$\lim_{n\to\infty} \mu_n = \mu_{opt}.$$

∎

This solves the $\mathcal{H}^2$ *Purely Finite Dimensional Sensitivity Problem.*
Proof An upper-semicontinuity result was established in the proof of theorem ??. To prove this theorem it suffices to prove a lower-semicontinuity result. The proof of this follows from the following inequality

$$\mu_{opt} \leq \left\|W(1 - P\tilde{Q}_n)\right\|_{\mathcal{H}^2} \leq \left\|W(1 - P_n\tilde{Q}_n)\right\|_{\mathcal{H}^2} + \left\|W(P_n - P)\tilde{Q}_n)\right\|_{\mathcal{H}^2}$$

where $\tilde{Q}$ is selected as in theorem 8.4.2. ∎

It should be emphasized that the above result has been obtained even though the *optimal performance* μ_{opt} need not be continuous with respect to perturbations in the plant P, even when the uniform topology is imposed [53]. Even if it were continuous, there are many plants in $\mathcal{H}^\infty$ which cannot be approximated by $R\mathcal{H}^\infty$ approximants; e.g. a delay (see proposition 2.10.1). Given this, it is also imperative to point out that the above result has been shown even though the approximants $\{P_n\}_{n=1}^{\infty}$ need not approximate the plant P uniformly.

The following corollary gives great insight into the computation of the optimal performance μ_{opt}.

Corollary 8.5.1 (Computation of Optimal Performance)

$$\lim_{n\to\infty} \left\|\Pi_{\mathcal{H}^{2\perp}} W P_{n_i}^*\right\|_{\mathcal{H}^2} = \mu_{opt}.$$

∎

Proof First we note that

$$\mu_{opt} = \left\|\Pi_{\mathcal{H}^{2\perp}} W P_i^*\right\|_{\mathcal{H}^2}.$$

This follows from simple analysis. The proof then follows from the inequality

$$\left|\left\|\Pi_{\mathcal{H}^{2\perp}} W P_{n_i}^*\right\|_{\mathcal{H}^2} - \left\|\Pi_{\mathcal{H}^{2\perp}} W P_i^*\right\|_{\mathcal{H}^2}\right| \le \left\|\Pi_{\mathcal{H}^{2\perp}} W (P_{n_i} - P_i)^*\right\|_{\mathcal{H}^2}$$

and lemma 2.10.2. ∎

Comment 8.5.1 (Implication)
The above corollary implies that the optimal performance μ_{opt} can be computed by solving a sequence of Lyapunov equations. We recall that if $F(s) = C(sI - A)^{-1}B \in R\mathcal{H}^2$, then

$$\|F\|_{\mathcal{H}^2} = \sqrt{trace\{B'MB\}}$$

where M is the unique symmetric solution to the Lyapunov equation

$$A'M + MA + C'C = 0.$$

M is the *observability grammian* of F.

∎

8.6 Solution to $\mathcal{H}^2$ Loop Convergence Sensitivity Problem

Thus far, we have given conditions under which the *expected performance* μ_n and the *actual performance* $\tilde{\mu}_n$ approach the *optimal performance* μ_{opt}. We now investigate the behavior of the actual loop shapes. More specifically, we would like to know in what sense, if any, does $\tilde{Q}_n$ converge to a near-optimal solution $\tilde{Q}_{opt}$. Since $\tilde{Q}_n$ and $\tilde{Q}_{opt}$ are based on "inner solutions" Z_n and Z_{opt}, it makes sense to investigate the convergence of Z_n to Z_{opt}. We start with the following assumption.

Theorem 8.6.1 (Solution to Loop Convergence Problem)
If $Z_{opt} \in \mathcal{H}^\infty$ is the solution to

$$\min_{Z\in\mathcal{H}^\infty} \|W - P_i Z\|_{\mathcal{H}^\infty} = \|W - P_i Z_{opt}\|_{\mathcal{H}^\infty},$$

then

$$\lim_{n\to\infty} \|Z_n - Z_{opt}\|_{\mathcal{H}^\infty} = 0.$$

∎

Proof First we note that

$$Z_n = \Pi_{\mathcal{H}^2} W P_{n_i}^* = \sum_{k=1}^{l} c_k \frac{P_{n_i}^*(-a_k)}{s + a_k}$$

and

$$Z_{opt} = \Pi_{\mathcal{H}^2} W P_i^* = \sum_{k=1}^{l} c_k \frac{P_i^*(-a_k)}{s + a_k}.$$

The result then follows since the a_k lie in a compact set and P_{n_i} uniformly approximates P on compact sets. ∎

Comment 8.6.1 (Convergence of Loop Shapes)
The above theorem implies that if proper care is taken, we can construct sequences of compensators which yield loop shapes which approach the optimal loop shape uniformly. The practical implications of this are obvious. This makes the *Approximate/Design* approach presented in this chapter a truely useful design tool.

∎

This concludes our discussion of the $\mathcal{H}^2$ *Loop Convergence Sensitivity Problem.*

8.7 Unstable Plants

In this section we exploited the fact that W was real rational. If P is unstable then we need to deal with additional factors D_p and D_k and N_{p_i}. How to deal with these additional terms is a topic for future research.

8.8 Mixed-Sensitivity and Super-Optimal Performance Criteria

Extension of the ideas to mixed-sensitivity and super-optimal performance criteria is straight forward.

8.9 Summary

In this chapter a solution was presented to the $\mathcal{H}^2$ *Approximate/Design Sensitivity Problem.* More specifically, it was shown how near-optimal finite dimensional strictly proper compensators could be constructed for an infinite dimensional plant, given a weighted $\mathcal{H}^2$ sensitivity design specification. It was shown that the construction could be carried out on the basis of finite dimensional solutions obtained from appropriately formulated finite dimensional $\mathcal{H}^2$ sensitivity problems. The finite dimensional problems which one solves are "natural" weighted $\mathcal{H}^2$ sensitivity problems obtained by replacing the infinite dimensional plant by "appropriately" chosen finite dimensional approximants.

It was shown that, in general, approximants based on "open loop intuition", rather than on the control objective, may yield compensators which do not even guarantee stability when used with the infinite dimensional plant. It was also shown how "appropriate" approximants could be constructed. The approximants obtained were constructed so that their imaginary axis pole-zero

structure would not drastically differ from that of the plant. The construction presented does not require sophisticated mathematics or software. It can be used by practicing engineers with little effort.

We also provided a solution to the *Purely Finite Dimensional $\mathcal{H}^2$ Sensitivity Problem.* Here, the issue of computing the optimal performance was addressed. It was shown that the optimal performance could be computed by solving a sequence of finite dimensional eigenvalue/eigenvector problems rather than the typical infinite dimensional eigenvalue/eigenfunction problems which appear in the literature. This makes a once difficult problem, almost trivial. Examples were given to illustrate this.

Finally, conditions were given under which the near-optimal finite dimensional loop shapes could converge to the optimal infinite dimensional loop shape uniformly on compact frequency intervals. This was illustrated in our solution to the $\mathcal{H}^2$ *Loop Convergence Sensitivity Problem.* This makes the Approximate/Design approach presented in the chapter a truely promising engineering tool.

Chapter 9

Summary and Directions for Future Research

9.1 Summary

In this thesis three new problems were formulated. They were the (1) *$\mathcal{N}$-Norm Approximate/Design J-Problem*, the (2) *$\mathcal{N}$-Norm Purely Finite Dimensional J-Problem*, and the (3) *$\mathcal{N}$-Norm Loop Convergence J-Problem.* Solutions were presented for $\mathcal{H}^\infty$ and $\mathcal{H}^2$ sensitivity and mixed-sensitivity performance criterion. We now summarize the results obtained for each problem and criterion.

It has been shown how one can systematically design near-optimal finite dimensional compensators for infinite dimensional plants, based on finite dimensional approximants. The criteria used to determine optimality are standard weighted $\mathcal{H}^\infty$ and $\mathcal{H}^2$ sensitivity and mixed-sensitivity performance measures.

More specifically, it has been shown that given an "appropriate" finite dimensional approximant for an infinite dimensional plant, one can solve a "natural" finite dimensional problem and modify the solution, using appropriate "roll-off" functions, in order to obtain a near-optimal, strictly proper, finite dimensional compensator. The "natural" problem is obtained by substituting the finite dimensional approximants in place of the infinite dimensional plant in the original optimization problem. The term "appropriate" was made precise and depended on the closed loop design objective.

For the $\mathcal{H}^\infty$ and $\mathcal{H}^2$ sensitivity problems, "appropriate" approximants were constructed by approximating the inner and outer parts of the plant separately. The inner part was approximated uniformly on compact frequency intervals and the outer part was approximated uniformly. The approximants of the outer part were constructed such that their zero structure did not differ drastically from that of the plant on the extended imaginary axis. By doing so, the inversion of the outer part of the approximants could be done uniformly and in a bounded manner. Such an inversion was necessary to insure stability. The need to approximate the inner and outer parts separately is tied to the fact that the optimal performance, when using a sensitivity performance criterion, is intimately dependent on the inner part of the plant. Such a criterion always require the inversion of the outer part.

For the $\mathcal{H}^\infty$ and $\mathcal{H}^2$ mixed-sensitivity problems, in which control action is penalized, it was shown that approximating the plant directly on compact sets would yield "appropriate" approximants. We emphasize that such approximants can be computed from frequency domain data.

This summarizes our results for the *$\mathcal{N}$-Norm Approximate/Design J-Problem* for $\mathcal{H}^\infty$ and $\mathcal{H}^2$ sensitivity and mixed-sensitivity criterion.

In addition, we showed that for the $\mathcal{H}^\infty$ problems, the optimal performance can be computed by solving a sequence of finite dimensional eigenvalue/eigenvector problems rather than the typical infinite dimensional eigenvalue/eigenfunction problem which appear in the literature. This could be done even in situations where the corresponding Hankel-Toeplitz operators were non-compact.

For the $\mathcal{H}^2$ problem, the optimal performance could be computed by solving sequences of finite dimensional Lyapunov equations in order to compute the $\mathcal{L}^2$ norms of real-rational functions associated with the approximants.

This summarizes our results for the *$\mathcal{N}$-Norm Purely Finite Dimensional J-Problem* for $\mathcal{H}^\infty$ and $\mathcal{H}^2$ sensitivity and mixed-sensitivity criterion.

For each performance criterion considered, conditions were given under which loop convergence could be guaranteed. That is, conditions under which the actual loop shapes approach the optimal loop shapes in a strong sense.

This summarizes our results for the *$\mathcal{N}$-Norm Loop Convergence J-Problem* for $\mathcal{H}^\infty$ and $\mathcal{H}^2$ sensitivity and mixed-sensitivity criterion.

In summary, our approach allows us to forgo solving a "complex" infinite dimensional $\mathcal{H}^\infty$ and $\mathcal{H}^2$ problems. It provides rigorous justification for some of the approximations that control engineers typically make in practice. In addition, it has been clearly demonstrated that approximants must be chosen on the basis of the closed loop control objective and not on the basis of "open loop" intuition.

9.2 Directions for Future Research

Obvious directions for future research include extensions to $\mathcal{L}^1$ performance criterion and to multivariable infinite dimensional systems. Such extensions are currently in progress.

In our sensitivity results, the inner and outer parts of the plant were approximates separately. Inner-outer factorizations may be difficult to obtain in practice. We would like to obtain conditions under which only the plant P must be approximated. This would enable us to generate designs on the basis of frequency response data.

Also associated with sensitivity paradigms are issues which arise when an infinite number of poles and zeros are present. Such issues must be carefully studied and understood.

Since our methods revolve around finite dimensional approximants, approximation algorithms must be developed, convergence rate experiments must be conducted, and covergence rate results need to be obtainded. These issues are currently being addressed by many researchers in both the mathematics and engineering disciplines.

Finally, we contend that the ideas presented extend to areas such as system identification and distribute control. What you make small in an identification algorithm should depend on the control objective; not on some stability proof. What knowlege local controllers should have depends on the global closed loop system objective.

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